6 FUZZY RELATIONS
AND FUZZY GRAPHS

6.1 Fuzzy Relations on Sets and Fuzzy Sets

Fuzzy relations are fuzzy subsets of $X \times Y$, that is, mappings from $X \rightarrow Y$. They have been studied by a number of authors, in particular by Zadeh [1965, 1971], Kaufmann [1975], and Rosenfeld [1975]. Applications of fuzzy relations are widespread and important. We shall consider some of them and point to more possible uses at the end of this chapter. We shall exemplarily consider only binary relations. A generalization to $n$-ary relations is straight forward.

**Definition 6–1**

Let $X, Y \subseteq R$ be universal sets then

$$\bar{R} = \{(x,y), \mu_R(x,y)\} \mid (x,y) \subseteq X \times Y\}

is called a fuzzy relation on $X \times Y$.

**Example 6–1**

Let $X = Y = R$ and $\bar{R}$: = “considerably larger than.” The membership
function of the fuzzy relation, which is, of course, a fuzzy set on \( X \times Y \) can then be:

\[
\mu_R(x, y) = \begin{cases} 
0 & \text{for } x \leq y \\
\frac{(x - y)}{10y} & \text{for } y < x \leq 11y \\
1 & \text{for } x > 11y
\end{cases}
\]

A different membership function for this relation could be

\[
\mu_R(x, y) = \begin{cases} 
0 & \text{for } x \leq y \\
(1 + (y - x)^{-2})^{-1} & \text{for } x > y
\end{cases}
\]

For discrete supports fuzzy relations can also be defined by matrices.

**Example 6–2**

Let \( X = \{x_1, x_2, x_3\} \) and \( Y = \{y_1, y_2, y_3, y_4\} \)

\[
\bar{R} = \text{"x considerably larger than y"} =
\begin{array}{cccc}
  & y_1 & y_2 & y_3 & y_4 \\
x_1 & 0 & 0 & .1 & .8 \\
x_2 & 0 & .8 & 0 & 0 \\
x_3 & .1 & .8 & 1 & .8
\end{array}
\]

and

\[
\tilde{Z} = \text{"y much bigger than x"} =
\begin{array}{cccc}
  & y_1 & y_2 & y_3 & y_4 \\
x_1 & .4 & .4 & .2 & .1 \\
x_2 & .5 & 0 & 1 & 1 \\
x_3 & .5 & .1 & .2 & .6
\end{array}
\]

In definition 6–1 it was assumed that \( \mu_R \) was a mapping from \( X \times Y \) to \([0,1]\), that is, it assigned to each pair \((x,y)\) a degree of membership in the unit interval. In some instances, such as in graph theory, it is useful to consider fuzzy relations which map from fuzzy sets contained in the universal sets into the unit interval. Then definition 6–1 has to be generalized [Rosenfeld 1973].