§14. FORCING AND MODEL COMPLETIONS

Let L be a language of fixed similarity type and let $\mathcal{M}$ be a fixed class of S-modal structures for ML. Let $\Phi$ be a fixed set of formulas of ML-$\mathcal{L}_s$ which contains all atomic formulas and is closed under subformulas. The S-forcing property $\mathcal{P}(\mathcal{M}, \Phi)$ is described as follows. Let $\Phi(C)$ be the set of all sentences of MK-$\mathcal{L}_s$ of the form $A_{x_1, \ldots, x_n}[c_1, \ldots, c_n]$ where $A \in \Phi$ and $c_1, \ldots, c_n \in C = \cup_{u \in U} C$. The set $P$ of conditions consists of all pairs $\langle \alpha, p \rangle$ meeting the following:

(i) $\alpha$ is a finite set of atomic sentences of LB$[U]$;
(ii) $p$ is a finite subset of $\Phi(C) \times U$;
(iii) there exist a structure $\mathcal{U} = \langle \mathcal{A}_k, K, R, O, N \rangle$ in $\mathcal{M}$ and maps $\delta: U \cup \{O^*\} \to K$ and $\pi: C \times U \to U(\mathcal{A})$ such that:
   (a) for each $c \in C^*$ and $u \in U \cup \{O^*\}$, $\pi(c, u) \in |\mathcal{A}_{\delta(u)}|$.
   (b) $\delta(O^*) = O$.
   (c) for each formula $B_{x_1, \ldots, x_n}[u_1, \ldots, u_n]$ in $\alpha$ where $B$ contains no elements of $U$, we have $b(\mathcal{U}) = B[\delta(u_1), \ldots, \delta(u_n)]$.
   (d) for each pair $\langle A_{x_1, \ldots, x_n}[c_1, \ldots, c_n], u \rangle$ in $p$, we have $\mathcal{U} \models_{\delta(u)} A[\pi(c_1, u), \ldots, \pi(c_n, u)]$.

We say that $\langle \delta, \pi \rangle$ satisfies $\langle \alpha, p \rangle$ in $\mathcal{U}$, and that $\langle \alpha, p \rangle$ is satisfiable in $\mathcal{U}$.

Define $\langle \alpha, p \rangle \leq \langle \beta, q \rangle$ iff $\alpha \subseteq \beta$ and $p \subseteq q$. Let $f(\langle \alpha, p \rangle)$ be the set of atomic sentences in $\alpha$, and for $u \in U$, let $h(\langle \alpha, p \rangle, u)$ be the set of atomic sentences $A$ such that $\langle A, u \rangle \in p$.

In the case that $\Phi$ consists precisely of all $\Diamond \Box$-formulas, we write $\mathcal{P}(\mathcal{M})$ for the forcing condition $\mathcal{P}(\mathcal{M}, \Phi)$. If $\mathcal{M}$ is the class of all S-modal models of the S-theory $T$, we say that $\mathcal{U}$ is $T$-generic if $\mathcal{U}$ is $\mathcal{P}(\mathcal{M})$-generic.

Now let $\langle \alpha, p \rangle$ be a condition in the forcing property $\mathcal{P}(\mathcal{M}, \Phi)$. An arbitrary S-modal structure $\mathcal{U}$ (not necessarily an element of $\mathcal{M}$) is said to be a model of $\langle \alpha, p \rangle$ at $w$ if there exists an interpretation $u^\mathcal{U}$ in $K$ of the constants $u$ of $U$, together with interpretations $c^\mathcal{U}_k$ of the constants $c \in C^*$ in each $|\mathcal{A}_k|$ for $u^\mathcal{U}_k = k \in K$ such that each formula of $\alpha$ is valid in $b(\mathcal{U})$ and for each formula $A$ of $p$, if $k = u^\mathcal{U}_k$, then $\mathcal{U} \models_k A$, where the constants $c$ of $A$ are interpreted as $c^\mathcal{U}_k$. Let $A \in \Phi(c)$. We will write $\langle \alpha, p \rangle \models \pi A$ if every model of $\langle \alpha, p \rangle$ is model of $A$ where the same interpretations of

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c∈C are used for both p and A. We will say that A∈Φ(C) is consistent with \langle x, p \rangle at u if \langle x, p \cup \{\langle A, u \rangle\} \rangle is satisfiable in some S-modal structure (not necessarily an element of \mathcal{M}).

**Lemma 14.1.** Let \langle x, p \rangle be a condition in the forcing property \mathcal{P}(\mathcal{M}, \Phi) and let A∈Φ(C). If \langle x, p \rangle \models_u A, then \langle x, p \rangle \models_u^* A, and if \langle x, p \rangle \not\models_u A, then A is consistent with \langle x, p \rangle at u.

**Proof:** We proceed by induction on the number of logical symbols in A.

**Case 1.** A is atomic. Then \langle x, p \rangle \models_u A if \langle \beta, q \rangle \models \langle x, p \rangle \exists \langle \gamma, r \rangle \geq \langle x, p \rangle \langle A, u \rangle \in r]. Taking \langle \beta, q \rangle = \langle x, p \rangle, there is a \langle \gamma, r \rangle \geq \langle x, p \rangle with \langle A, u \rangle \in r. But since \langle \gamma, r \rangle is a condition, there is in fact an \mathcal{U} \in \mathcal{M} and interpretations in \mathcal{U} witnessing that \langle \gamma, r \rangle is consistent and a fortiori, that A is consistent with \langle x, p \rangle at u. Now suppose that not \langle x, p \rangle \models_u^* A, so that

\exists \langle \beta, q \rangle \geq \langle x, p \rangle \forall \langle \gamma, r \rangle \geq \langle x, p \rangle \langle A, u \rangle \in r].

By Lemma 13.7, there is a \langle \gamma, r \rangle \geq \langle x, p \rangle such that \langle x, p \rangle, w \in r. Since \langle \gamma, r \rangle is a condition, there is an \mathcal{U} \in \mathcal{M} which is a model of r, and therefore also of \neg A at w. But \langle x, p \rangle \leq \langle \gamma, r \rangle, so \mathcal{U} is also a model of \langle x, p \rangle at u and so by hypothesis, \mathcal{U} is a model of A at u, a contradiction. Thus we must have \langle x, p \rangle \models_u A.

**Case 2.** A is \neg B. Suppose \langle x, p \rangle \models_u \neg B and let \langle \beta, q \rangle \geq \langle x, p \rangle. Then \langle \beta, q \rangle \models_u \neg B. Hence B is not consistent with \langle \beta, q \rangle at u, and so by induction, \neg \langle \beta, q \rangle \models_u B. It follows that for some \langle x, p \rangle \langle A, u \rangle \in r, \langle \gamma, r \rangle \models_u A. Hence \langle x, p \rangle \models_u \neg B. Next assume that \langle x, p \rangle \models_u \neg B and \langle \beta, q \rangle \models \langle x, p \rangle. Then \langle \beta, q \rangle \models_u \neg B, so \neg \langle \beta, q \rangle \models_u B. By the above, then \neg \langle \beta, q \rangle \models_u B, and since \langle \beta, q \rangle is a condition, it follows that B is consistent with \langle \beta, q \rangle at u, and hence B is consistent with \langle x, p \rangle at u.

**Case 3.** A is \lor \Psi. Suppose that \langle x, p \rangle \models_u \lor \Psi, then and let \langle \beta, q \rangle \geq \langle x, p \rangle. Then \langle \beta, q \rangle \models_u \lor \Psi. Since \langle \beta, q \rangle is satisfiable in \mathcal{M}, then for some B∈\Psi, \langle \beta, r \rangle = \langle \beta, q \cup \{\langle B, u \rangle\} \rangle is satisfiable in \mathcal{M}. Since B∈Φ(C) then \langle \beta, r \rangle is a condition and \langle \beta, r \rangle \geq \langle \beta, q \rangle. Since \langle \beta, r \rangle \models_u B, then by induction \langle \beta, r \rangle \models_u^* B, and hence for some \langle \gamma, s \rangle \geq \langle \beta, r \rangle \geq \langle \beta, q \rangle, \langle \gamma, s \rangle \models_u^* B, and so \langle \gamma, s \rangle \models_u \lor \Psi. Hence \langle x, p \rangle \models_u \lor \Psi.

Now suppose that \langle x, p \rangle \models_u \lor \Psi, so that for some \langle \beta, q \rangle \geq \langle x, p \rangle, \langle \beta, q \rangle \models_u \lor \Psi and hence for some B∈\Psi, \langle \beta, q \rangle \models_u B, and therefore \langle \beta, q \rangle \models_u^* B. By induction, B is consistent with \langle \beta, q \rangle at u and since \langle x, p \rangle \leq \langle \beta, q \rangle, then B is consistent with \langle x, p \rangle at u.

**Case 4.** A is \exists x B. Suppose that \langle x, p \rangle \models_u \exists x B and that \langle \beta, q \rangle \geq \langle x, p \rangle.