A NEW MATRIX DECOMPOSITION FOR SIGNAL PROCESSING

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ABSTRACT. We extend the generalized singular value decomposition to a new decompo­
sition that can be updated at a low cost. In addition, we show how a forgetting factor can
be incorporated in our decomposition.

KEYWORDS. ULV decomposition, generalized SVD, updating, forgetting factor, signal
processing.

1 Problem definition

A recurring matrix problem in signal processing concerns generalized eigenvalues:

\[ A^T A x = \lambda B^T B x, \]

where \( A \) is \( n \times p \), \( n \geq p \), \( B \) is \( m \times p \), and \( m \geq p \). We assume further that the matrix \( B \) has
full column rank. Often, the generalized eigenvalues, call them \( d_j^2 \)'s, satisfy this property:

\[ d_1^2 \geq d_2^2 \geq \cdots \geq d_{p-k}^2 \gg d_{p-k+1}^2 \approx \cdots \approx d_p^2. \]

The \( k \)-dimensional subspace spanned by the eigenvectors corresponding to the \( k \) smallest
generalized eigenvalues is called the noise subspace. We are interested in the following
problem.

Noise Subspace Problem. Compute an orthonormal basis for the noise subspace.
2 ULLV decomposition

This problem has a known solution for the special case where \( B = I_p \), where \( I_p \) denotes a \( p \times p \) identity matrix. Compute a singular value decomposition (SVD) of \( A \):

\[
A = U D_A V^H,
\]

where \( U \) is \( n \times p \) and orthonormal, i.e., \( U^H U = I_p \), \( V \) is \( p \times p \) and unitary, \( D_A \) is diagonal and \( D_A = \text{diag}(d_1, \ldots, d_p) \). From (1) we get

\[
d_1 \geq d_2 \geq \cdots \geq d_{p-k} \gg d_{p-k+1} \approx \cdots \approx d_p \geq 0,
\]

and the desired orthonormal basis is given by the last \( k \) columns of \( V \). However, the SVD is not amenable to efficient updating when a new row is added to \( A \). A clever procedure was devised by Stewart [2] in the form of the ULV decomposition (ULVD):

\[
A = U L_A V^H,
\]

where \( U \) is orthonormal and \( V \) unitary as in the SVD, but the middle matrix \( L_A \) is lower triangular and essentially block diagonal. In particular,

\[
L_A = \begin{pmatrix}
\hat{L}_A & 0 \\
E & K
\end{pmatrix},
\]

where

(i) \( \hat{L}_A \) and \( K \) are lower triangular and \( \hat{L}_A \) is \( (p-k) \times (p-k) \);

(ii) \( \sigma_{\min}(\hat{L}_A) \approx d_{p-k} \) and \( ||E||^2 + ||K||^2 \approx d^2_{p-k+1} + \cdots + d^2_p \).

By \( \sigma_{\min}(M) \) we mean the smallest singular value of a matrix \( M \), and by \( ||M|| \) we refer to the Frobenius norm of the matrix \( M \). Essentially, Stewart showed that to separate out the noise subspace from the signal subspace, it suffices to reduce \( A \) to the 2 \( \times \) 2 block lower triangular form \( L_A \), where both \( E \) and \( K \) are very small in norm. The last \( k \) columns of \( V \) provide an orthonormal basis for the noise subspace.

In this paper we consider the noise subspace problem for the general case where \( B \neq I_p \). First, the problem may be solved via the generalized SVD (GSVD):

\[
A = U_A D_A L V^H \quad \text{and} \quad B = U_B L V^H,
\]

where \( U_A \) is \( n \times p \) and orthonormal, \( U_B \) is \( m \times p \) and orthonormal, \( V \) is \( p \times p \) and unitary, \( L \) is \( p \times p \) and lower triangular, and \( D_A = \text{diag}(d_1, \ldots, d_p) \). If the generalized singular values \( d_j \)'s satisfy (2), then the last \( k \) columns of \( V \) provide a basis for the noise subspace.

We propose here a generalized ULVD (ULLVD):

\[
A = U_A L_A L V^H \quad \text{and} \quad B = U_B L V^H,
\]

where \( U_A, U_B, V \) and \( L \) are just as in the GSVD. The new middle matrix \( L_A \) has the same form as in (3) and the desired orthonormal basis is given by the last \( k \) columns of \( V \).