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A LOGIC WITH PROGRESSIVE TENSES

1. INTRODUCTION

Let \((W,<)\) be a strictly linearly ordered set (a "time-line"). Having a Kripke model over \((W,<)\) we interpret simple past and future operators in the well-known way:

\[ x \models FA \iff (\exists y > x) y \models A \]
\[ x \models PA \iff (\exists y < x) y \models A. \]

In [1] Dana Scott proposed the following semantics for the present progressive operator ("it is being the case that..."):

\[ x \models \Box A \iff \exists I (I \text{ is an open interval } \land \]
\[ x \in I \land \forall z (z \in I \implies z \models A). \]

(As usual, open intervals can be of four types: \((y,z) = \{t \mid y < t < z\}, <(y) = \{t \mid y < t\}, >(y) = \{t \mid y > t\}, W\).) Certainly this is nothing but neighbourhood semantics corresponding to the interval topology in \((W,<)\) (whose open sets are arbitrary unions of open intervals).

As usual, the modal logic of the time-line \((W,<)\) is the set of all formulas valid in \((W,<)\) (i.e., of those true in any world of any Kripke model over \((W,<)\)). Such a logic will be denoted by \(L^{FF}(W,<)\) (\(L^{\Box}(W,<)\) denotes its fragment containing only formulas without \(F\) and \(P\)).

Due to results of McKinsey and Tarski [2] some logics \(L^{\Box}(W,<)\) can be easily axiomatized. Namely [2] proves that the modal logic of any dense-in-itself metrizable topological space with a countable base (considered as a neighbourhood frame) is just \(S4\). In particular, \(L^{\Box}(W,<) = S4\) for any dense \((W,<)\) which can be embedded into \((\mathbb{R},<)\).

From results of Abashidze [3] it follows that \(L^{\Box}(\omega^\omega,<) = Grz\) (Grzegorczyk logic), and for any ordinal \(\alpha\) such that \(\omega^{n-1} \leq \alpha < \omega^n\), \(L^{\Box}(\alpha,<) = Grz + B_\alpha\), the least extension of \(Grz\) lying in the
n-th slice. This is all we know about logics $L^\square(W, <)$. A complete
description of all logics of this type is an open problem.

Axiomatizing of logics $L^{FP\square}(W, <)$ turns out to be a more dif-
ficult task. For example, what can we say about $FP\square$-logics of ra-
tionals or of reals? By a well-known procedure, $FP\square$-formulas are
translated into universal monadic (second-order) formulas. Viz., $A^+$, the translation of $A$, is the universal closure of the monadic
formula $A^*$; and $A^*$ is defined inductively as follows.

\begin{align*}
(p_n)^* &= x \in X_n \ (p_n \text{ is a propositional variable}, \\
& \hspace{1cm} x \text{ is an individual variable}, \\
& \hspace{1cm} X_n \text{ is a set variable}), \\
(A \lor B)^* &= A^* \lor B^*, \\
(\neg A)^* &= \neg A^*, \\
(FA)^* &= \exists y (x < y \land (y/x)A^*), \\
(PA)^* &= \exists y (y < x \land (y/x)A^*), \\
(\Box A)^* &= \exists y_1 \exists y_2 (y_1 < x \land x < y_2 \land \\
& \hspace{1cm} \forall z (y_1 < z \land z < y_2 \supset (z/x)A^*)) \lor \\
& \exists y (y < x \land \forall z (y < z \supset (z/x)A^*)) \lor \\
& \exists y (x < y \land \forall z (z < y \supset (z/x)A^*)) \lor \\
& \forall x A^*.
\end{align*}

Then the validity of $A$ in $(W, <)$ is equivalent to the truth of $A^+$ (in the classical sense), and $L^{FP\square}(W, <)$ is embedded into
the universal monadic theory of $(W, <)$. Gurevich and Burgess
proved that such theories of $(\mathbb{R}, <)$ and of $(\mathbb{Q}, <)$ are decidable [4].
Thus corresponding logics are decidable as well.

But this result does not provide us any concrete axiomatiza-
tion. Furthermore, the proof in [4] appeals to Rabin's theorem
[5], so the decidability algorithm is very complicated (non-Kalmar
elementary) [9].

Nevertheless, using traditional methods of modal logic we can
present a finite axiomatization and a simpler deciding procedure
for $L^{FP\square}(\mathbb{Q}, <)$.