

HIGHER GRADE MATERIAL STRUCTURES

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1. Introduction

It has been often pointed out, by critics and supporters alike, that the theory of inhomogeneities of Noll [1] and Wang [2] does not enjoy the generality often demanded by those propagating the so-called lattice model. This is because in the structural approach to the theory of continuous distribution of defects it has been suggested that, although the presence of dislocations shows through the non-vanishing torsion of the material connection, disclinations are measured by the curvature of such a connection; see e.g. Anthony [3]. Since any constitutive functional associated with a simple elastic material body induces, by definition, a locally integrable parallelism it appears that the disclinations, and possibly other defects, are ruled out. A structural approach suggests also that bodies with defects, disclinations in particular, are subject to multipolar stresses. Thus, it seems natural to investigate the possibility of describing disclinations in the realm of the higher-grade materials as originally suggested by Elzanowski and Epstein [4].

In this short note we sketch the mathematical foundations of the theory of continuous distributions of defects of the second-grade elastic body. We point also at possible generalizations of this approach to the higher grade cases and the materials with microstructures.

2. Uniform Elastic Materials

Let the body \mathbf{B} be represented by a smooth, orientable, connected and boundary-less differentiable manifold over \mathbb{R}^3 . Suppose, for the simplicity and the clarity of this presentation, that it can be covered by a global chart. Hence, we choose one

such chart $\psi_0 : \mathbf{B} \rightarrow \mathbb{R}^3$ as the reference configuration of \mathbf{B} and identify the body with its image $\psi_0(\mathbf{B}) \subset \mathbb{R}^3$.

Given any other configuration $\phi : \mathbf{B} \rightarrow \mathbb{R}^3$, $\phi \circ \psi_0^{-1} : \psi(\mathbf{B}) \rightarrow \phi(\mathbf{B})$ is called the *deformation* of \mathbf{B} . A k -jet of such a deformation at $x \in \mathbf{B}$ is called a *local configuration of order k* of the material point x . We assume that the mechanical properties of \mathbf{B} are completely characterized by a smooth real-valued function \mathcal{W} on the space of all local configurations of \mathbf{B} , say $H^k(\mathbf{B})$. In the mathematical literature the space $H^k(\mathbf{B})$ is known as the bundle of holonomic k -frames of \mathbf{B} ; see e.g. Cordero et al [5]. Its structure group G^k is the semidirect product of the general linear group $GL(3, \mathbb{R})$ and the space of all multilinear symmetric \mathbb{R}^3 -valued forms of order 2 through k , denoted as N_1^k . We say that the material body \mathbf{B} characterized by the strain energy function \mathcal{W} is *smoothly uniform*, i.e. built of the same material points, if there exists a transitive (over \mathbf{B}) pseudo-group of smooth automorphisms $\mathcal{P} : H^k(\mathbf{B}) \rightarrow H^k(\mathbf{B})$ such that

$$\mathcal{W}(\mathcal{P}(p^k)) = \mathcal{W}(p^k) \quad (1)$$

for every local configuration $p^k \in H^k(\mathbf{B})$. Equivalently, \mathbf{B} is smoothly uniform if there exists a section $\mathfrak{l} : \mathbf{B} \rightarrow H^k(\mathbf{B})$, called the *material reference*, such that for any $x, y \in \mathbf{B}$ and any automorphism \mathcal{P} of equation (1)

$$\mathcal{P}(\mathfrak{l}(x)) = \mathfrak{l}(y). \quad (2)$$

Note that the strain energy function \mathcal{W} may have, over a given material point, a non-trivial isotropy group. In fact, it may have different isotropy groups over different base points. If however the body \mathbf{B} is uniform the equation (1) implies that these isotropy groups are isomorphic. If the uniform \mathcal{W} has a nontrivial isotropy group the choice of the uniform reference is not unique. Indeed, any gauging of the uniform reference by the isotropy group (the symmetry group of any and so all material points of \mathbf{B}) produces another uniform reference. We eliminate this nontrivial degree of freedom of choice by assuming that the symmetry group of any material point of \mathbf{B} is at the most a discrete subgroup of the general linear group $GL(3, \mathbb{R})$. This guarantees, due to the smoothness of \mathfrak{l} , the uniqueness of the material reference.

The material section \mathfrak{l} induces on $H^k(\mathbf{B})$ the so-called *material parallelism* by lifting the tangent space of \mathbf{B} to the frame bundle. There exists also, as shown by Elżanowski and Prishepionok [6], a smooth function $\tilde{\mathcal{W}} : G^k \rightarrow \mathbb{R}$ satisfying the equation

$$d\mathcal{W}(\mathfrak{l}(\pi(p^k))g)(\xi) = d\tilde{\mathcal{W}} \circ d\mathfrak{A}_{g^{-1}} \circ \omega^k(p^k)(\xi) \quad (3)$$