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GRASSMANN'S DIALECTICS AND CATEGORY THEORY

PROGRAMMATIC OUTLINE

In several key connections in his foundations of geometrical algebra, Grassmann makes significant use of the dialectical philosophy of 150 years ago. Now, after fifty years of development of category theory as a means for making explicit some nontrivial general arguments in geometry, it is possible to recover some of Grassmann's insights and to express these in ways comprehensible to the modern geometer. For example, the category $\mathcal{A}$ of affine-linear spaces and maps (a monument to Grassmann) has a canonical adjoint functor to the category of (anti)commutative graded algebras, which as in Grassmann's detailed description yields a sixteen-dimensional algebra when applied to a three-dimensional affine space (unlike the eight-dimensional exterior algebra of a three-dimensional vector space). The natural algebraic structure of these algebras includes a boundary operator ∂ which satisfies the (signed) Leibniz rule; for example, if A, B are points of the affine space then the product AB is the axial vector from A to B which the boundary degrades to the corresponding translation vector: $\partial(AB) = B-A$ (since $\partial A = \partial B = 1$ for points). Grassmann philosophically motivated a notion of a "simple law of change," but his editors in the 1890's found this notion incoherent and decided he must have meant mere translations. However, translations are insufficient for the foundational task of deciding when two formal products are geometrically equal axial vectors. But if "law of change" is understood as an action of the additive monoid of time, "simple" turns out to mean that the action is internal to the category $\mathcal{A}$ at hand, in the following sense: The affine category has a unique closed structure, consisting of an internal hom functor with an adjoint tensor-product functor which neither is the cartesian product nor underlies the usual tensor product of vector spaces, and since addition of times does define an internal monoid structure $R \otimes R \rightarrow R$ with respect to this tensor product, it is very natural to interpret "internal law of becoming" on a space E to mean an action (= "flow") $R \otimes E \rightarrow E$ with respect to this tensor. Such actions

G. Schubring (ed.), Hermann Günther Graßmann (1809–1877): Visionary Mathematician, Scientist and Neohumanist Scholar, 255–264.
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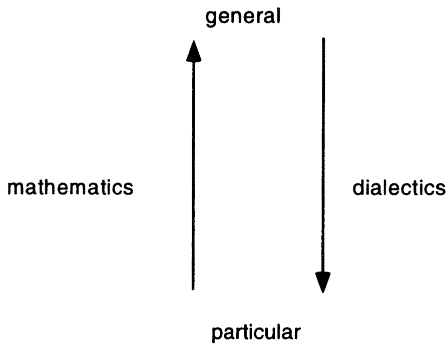
turn out to be determined by shear transformations, of which there are indeed enough to detect equality of axial vectors!

ORIENTATION

Grassmann in his philosophical introduction describes the two-fold division of formal sciences, that is, the science of thinking, into dialectics and mathematics. He briefly describes dialectics as seeking the unity in all things, and he describes mathematics as the art and practice of taking each thought in its particularity and pursuing it to the end. There is a need for an instrument which will guide students to follow in a unified way both of these activities, passing from the general to the particular and from the particular to the general.



I believe that the theory of mathematical categories (which was made explicit 50 years ago by Eilenberg and Mac Lane, codifying extensive work done by Hurewicz in particular during the 1930's), can serve as such an instrument. It was introduced and designed in response to a very particular question involving passage to the limit in calculating cohomology of certain portions of spheres, but this particular calculation necessitated an explicit recognition of the manner in which these spaces were related to all other spaces and, in particular, how their motion might induce other motions. In other words, category theory was introduced (and still serves) as "a universal geometrical calculus."

Looking more closely into Grassmann, Stephen Schanuel and I found numerous ways in which it could be justified to say that Grassmann was a pre-cursor of category theory. The general algebraic operations which he discussed have become the explicit object of a particular developed theory, and those general concepts, general operations, systems of operations and systems of equations in invariant coordinate free form have been made into a part of category theory. More specifically, we find that in certain cases the famous distinction between analytic and synthetic operations can only be explained in terms of adjoint functors.