Chapter 12

Singularities and the Nowhere Dense Algebras of Generalised Functions

Let us present certain explicit details about why in this work, we use the nowhere dense differential algebras (1.6), (7.1.4), (7.1.3). We shall do so along two lines:

First, we shall shortly recall the very reasons, as well as the way the nowhere dense differential algebras were first constructed and also found useful, back in the mid 1970s, see Rosinger [3-5].

Then, we shall further elaborate on the recent interest in their use in dealing with large classes of singularities in abstract differential geometry, and in particular, de Rham cohomology and general relativity.

The question about the role of the nowhere dense differential algebras - regardless of the fact that they were the first to be used in any systematic nonlinear theory of generalised functions - is indeed a question which may quite legitimately arise, in view of the following two facts:

First, as mentioned earlier, in Rosinger [6], a wide ranging purely algebraic characterization was given for all possible differential algebras of generalised functions which contain the Schwartz distributions. And this characterization, together with several examples in Rosinger [6], show that there are infinitely many rather different classes of such differential algebras. Moreover, based solely on their respective algebraic constructions, there is no a priori way to determine among them certain canonical algebras. Indeed, when seen from a purely theoretical, algebraic point of view, it is clear that one cannot choose a distinguished or privileged algebra, or class of such algebras. This phenomenon is directly related to the quotient structure of such algebras, and it depends on a very simple algebraic, more precisely, ring theoretic fact, which shows that the ideals used in such quotient constructions do not, in purely algebraic terms, allow us to distinguish any particular one among...
them, see Rosinger [6, pp. 118,119], [8, pp. 98,99]. This situation is quite similar with, and in fact it is in a way an extension of, the well known fact that, when constructing various instances of the nonstandard reals $^\ast \mathbb{R}$, there is no canonical way for choosing the respective ultra filters.

Second, the later introduced, Colombeau type algebras, one of the types included in the mentioned algebraic characterization, see Rosinger [7], have recently known a certain popularity in a number of applications developed by a variety of researchers, and related to generalised or numerical solutions of nonlinear PDEs, stochastic PDEs, or general relativity, see the earlier mentioned literature, while for stochastic applications, see Albeverio, et., al., Oberguggenberger [2], Oberguggenberger & Russo, Russo, Martias [1,2], Lozanov Crvenkovic & Pilipovic.

As mentioned, in order to answer the above question related to our choice of the nowhere dense differential algebras of generalised functions, we shall give certain details about the original reasons underlying their construction. Then as well, we shall further indicate some of their recent uses which go beyond the initial range of applications developed in relation to solving nonlinear PDEs, applications which, this time, reach as far as recent joint work with A Mallios, on de Rham cohomology in abstract differential geometry, see Mallios & Rosinger.

And now, to the original reasons for constructing the nowhere dense differential algebras of generalised functions.

**Generalised Functions Means Dealing With Singularities.** It is rather instructive to see how the issue of singularities of functions happened to become particularised, and thus to some extent marginalised, in various theories of generalised functions, although quite clearly, it had at the beginning been the major reason for the emergence of these theories.

Indeed, the Sobolev and Schwartz linear version of the theory of generalised functions was, as is well known, developed in order to deal with nonclassical solutions of linear PDEs. And nonclassical, in this context, means solutions which are not smooth enough, that is, their order of smoothness is less than the order of partial differentiations in the respective PDEs.

In this way, a new concept of *singularity* was introduced and dealt with in the study of solutions of PDEs, namely, given by the subsets $\Gamma$ of the domain $\Omega$ of the respective PDEs, subsets on which the solutions failed to be smooth enough. Indeed, the earlier concept of singularity was typically related only to such points or subsets in the neighbourhood of which the solutions became unbounded. One notable exception to this classical view of singularities was that of the Riemann solvers of the nonlinear shock wave equation, which proved to point early towards the new concept of singularity.

The interest in the new type of singularity first appeared explicitly in the late 1920s, within a simple linear framework, in the case of the so called Dirac delta function $\delta$, although earlier, towards the end of the 1800s, Heaviside had encountered a similar phenomenon in his operational calculus aimed at solving constant coefficient ODEs.