11. Many-valued logics

The same thing cannot at the same time both belong and not belong to the same object and in the same respect. [ . . . ] Of any object, one thing must be either asserted or denied.

Aristotle

Logic changes from its very foundations if we assume that in addition to truth and falsehood there is also some third logical value or several such values.

Jan Łukasiewicz

The study of many-valued logic was initiated by Jan Łukasiewicz around 1920. He started with a three-valued logic, introducing in particular an implication for it (see [Łukasiewicz 1920, 1930] and [Łukasiewicz & Tarski 1930], a selection of Łukasiewicz’s papers can be found in [Borkowski 1970]). Another important step was the study of intuitionistic logic in [Heyting 1930, Gödel 1932], significant contributions were also made in, e.g., [Post 1921], [Moisil 1940, 1941] and [McNaughton 1951, Chang 1958, Rose & Rosser 1958] (for surveys on many-valued logics we refer to the monographs [Rosser & Turquette 1952, Rescher 1969] and [Gottwald 1989, 200x]).

For several decades, many-valued logic was considered a purely theoretical topic. The introduction of fuzzy sets in [Zadeh 1965] and, subsequently, of fuzzy logics (see, e.g., [Goguen 1969, Chang & Lee 1975], [Pavelka 1979a, 1979b, 1979c], [Novák 1987], [Hájek 1995b, 1998b]), and [Novák et al. 1999, Turunen 1999b, Cignoli et al. 2000] produced a new impact to the investigation of many-valued logics, in particular to [0,1]-valued logics based on t-norms, which are often called fuzzy logics.

Among various approaches to fuzzy logics, we describe two of them (studied in [Giles 1976, Hájek et al. 1996, Hájek 1998b, Esteva et al. 200x], on the one hand, and in [Butnariu et al. 1995], on the other hand (for a comparison see [Klement & Navara 1999b]) which are constructed in a similar way. Although starting from different basic logical connectives, they both use interpretations based on continuous t-norms (quite often even based on Frank t-norms, see Section 4.4). Different interpretations of the implication lead to different axio-
matizations, but most logics presented here are complete. A comparison of the properties, advantages and disadvantages of the two approaches will be given.

In order to keep this chapter readable it was impossible to describe in detail all important approaches to \([0, 1]\)-valued logics which were presented in the literature. For instance, the concepts introduced and studied in [Pavelka 1979a, 1979b, 1979c], [Höhle 1994, 1995] and [Novák et al. 1999] (see also [Höhle & Klement 1995]) have been omitted here.

11.1 Interpretations of connectives in fuzzy logics

A many-valued propositional logic in which the class of truth values is modeled by the unit interval \([0, 1]\), and which forms an extension of the classical Boolean logic, i.e., the two-valued logic with truth values \{0, 1\}, is quite often called a fuzzy logic ([Gottwald 1989, 200x] and [Hájek 1995b, 1998b]). In such a logic, the conjunction is usually interpreted by a triangular norm.

11.1 Definition

A (propositional) fuzzy logic is described as an ordered pair \( \mathcal{P} = (\mathcal{L}, \Omega) \) of a language (syntax) \( \mathcal{L} \) and a structure (semantics) \( \Omega \) specified as follows:

(i) The language of \( \mathcal{P} \) is a pair \( \mathcal{L} = (A, \mathcal{C}) \), where \( A \) is a finite or countably infinite set of atomic symbols and \( \mathcal{C} \) is a tuple of connectives.

(ii) The structure of \( \mathcal{P} \) is a pair \( \Omega = ([0, 1], \mathcal{M}) \), where \([0, 1]\) is the set of truth values, and the tuple \( \mathcal{M} \) consists of the interpretations (meanings) of the connectives in \( \mathcal{C} \).

For simplicity, we always consider the same set \( A \) of atomic symbols, i.e., all fuzzy logics have the same syntax, they may differ only by their semantics.

11.2 Definition

The class \( \mathcal{F}_\mathcal{P} \) of well-formed formulae in a fuzzy logic (\( \mathcal{P} \)-formulae for short) is defined inductively as follows:

(i) Each atomic symbol \( p \in A \) is a \( \mathcal{P} \)-formula.

(ii) If \( \Diamond \) is an \( n \)-ary connective and if \( \varphi_1, \varphi_2, \ldots, \varphi_n \) are \( \mathcal{P} \)-formulae, then \( \Diamond(\varphi_1, \varphi_2, \ldots, \varphi_n) \) is a \( \mathcal{P} \)-formula.

For each function \( t : A \rightarrow [0, 1] \), which assigns a truth value to each atomic formula, there exists always a unique natural extension of \( t \) to a truth assignment \( \overline{t}_\mathcal{P} : \mathcal{F}_\mathcal{P} \rightarrow [0, 1] \) which, for each atomic symbol \( p \), for each \( n \)-ary connective \( \Diamond \), its interpretation Meaning\(_\Diamond\) and for all \( \mathcal{P} \)-formulae \( \varphi_1, \varphi_2, \ldots, \varphi_n \), is obtained by induction in the following canonical way:

\[
\overline{t}_\mathcal{P}(p) = t(p), \\
\overline{t}_\mathcal{P}(\Diamond(\varphi_1, \varphi_2, \ldots, \varphi_n)) = \text{Meaning}\(_\Diamond\)(\overline{t}_\mathcal{P}(\varphi_1), \overline{t}_\mathcal{P}(\varphi_2), \ldots, \overline{t}_\mathcal{P}(\varphi_n)).
\]