Chapter VI

APPLICATIONS OF ZAGIER’S FORMULA (II)

In this chapter we present some recent results from [Fox, Urbanowicz and Williams, 1999]. Let \( d \) denote the discriminant of a quadratic field. Let \( n \) be the number of distinct prime factors of \( d \). Recall that \( \chi_d \) and \( h(d) \) denote the character and class number of the field respectively. Recall also that \( B_{k,\chi_d} \) denotes the generalized Bernoulli number attached to \( \chi_d \). It is shown in an elementary manner how Gauss’ congruence for imaginary quadratic fields \( h(d) \equiv 0 \pmod{2^n - 1} \) (see Chapter II) can be deduced from Dirichlet’s formula for \( h(d) \) (see Chapter I). We also generalize the Gauss congruence to 2-integral rational numbers \( (B_{k,\chi_d}/k) \). We prove that \( (B_{k,\chi_d}/k) \equiv 0 \pmod{2^n - 1} \) if \( \chi_d(-1) = (-1)^k \). This is a further application of Zagier’s identity.

1. PRELIMINARIES

1.1 Some Elementary Observations

As a consequence of his theory of genera for imaginary quadratic fields, Gauss obtained algebraically the congruence

\[ h(d) \equiv 0 \pmod{2^n - 1}, \tag{1} \]

where \( n \) is the number of distinct prime factors of \( d \) \( (d < 0) \), see section 8.5 in Chapter I. Dirichlet showed analytically that

\[ h(d) = \frac{w(d)}{2(2 - \chi_d(2))} \sum_{r=1}^{\lfloor |d|/2 \rfloor} \chi_d(r) \]  \tag{2}

(see formula (17) of Chapter I). We show in an elementary manner how Dirichlet’s formula (2) can be made to yield Gauss’ congruence (1). We accomplish
this by putting (2) into a form (see Theorem 65) from which (1) can be deduced by induction on $n$. The proof of Theorem 65 is based on three elementary lemmas. The first gives a congruence modulo a power of 2 for $\phi(|d|)$, where $\phi$ is Euler’s phi function. The second evaluates a sum which occurs in the proof of Theorem 65. The third puts (2) into a more general form for use in the proof of Theorem 65. Before the lemmas we give some elementary observations. The detailed proofs of the lemmas are left to the reader.

As $d$ is the discriminant of a quadratic field, we have $d \equiv 1 \pmod{4}$, $d \equiv 8 \pmod{16}$ or $d \equiv 12 \pmod{16}$. Moreover, we have

$$d = \prod_{p \mid d} p^*, \quad p \text{ prime}$$

where the prime discriminant $p^*$ corresponding to the prime $p \mid d$ is given by

$$p^* = (-1)^{(p-1)/2} p,$$

if $p$ is odd, and

$$2^* = \begin{cases} 8, & \text{if } d \equiv 8 \pmod{32}, \\ -8, & \text{if } d \equiv 24 \pmod{32}, \\ -4, & \text{if } d \equiv 12 \pmod{16}. \end{cases}$$

Write $2^* = 1$ if $d \equiv 1 \pmod{4}$. If $d < 0$, we have

$$|d/2^*| \equiv \begin{cases} 3 \pmod{4}, & \text{if } d \equiv 1 \pmod{4} \text{ or } d \equiv 8 \pmod{32}, \\ 1 \pmod{4}, & \text{if } d \equiv 12 \pmod{16} \text{ or } d \equiv 24 \pmod{32}. \end{cases}$$

Let $u$ denote the number of distinct prime divisors of $d$ which are congruent to 1 modulo 4 and $v$ the number of distinct prime divisors of $d$ which are congruent to 3 modulo 4, so that

$$u + v = \begin{cases} n, & \text{if } d \equiv 1 \pmod{4}, \\ n - 1, & \text{if } d \equiv 0 \pmod{4}, \end{cases}$$

and

$$v \equiv \begin{cases} 1 \pmod{2}, & \text{if } d \equiv 1 \pmod{4} \text{ or } d \equiv 8 \pmod{32}, \\ 0 \pmod{2}, & \text{if } d \equiv 12 \pmod{16} \text{ or } d \equiv 24 \pmod{32}. \end{cases}$$

### 1.2 Three Lemmas

We now prove the three elementary lemmas mentioned above.

**Lemma 1** ([Fox, Urbanowicz and Williams, 1999, Lemma 1]) *Let $d$ be the discriminant of an imaginary quadratic field. Let $n$ denote the number of*