INTRODUCTION

During the past decades several attempts have been made to reconsider the development of mathematics or the growth of mathematical knowledge from some unifying perspectives. Lakatos’ notion of “research programs” represents one such attempt. He did not find followers to pursue the matter beyond the few, and rather narrow, case studies he himself considered. A more recent attempt, still under discussion, was provoked by Kuhn’s work and led to the Crowe-Dauben debate on revolutions in mathematics.

In studying these papers, it occurred to me that it might be fruitful to consider controversies among influential mathematicians on matters of general interest to the scientific communities of their times. It turned out that such debates provided insights concerning the growth of mathematical knowledge, how it developed, when and why it changed its main direction, and why branches that had previously appeared lively ultimately withered. One advantage of studying such controversies is that they are usually well documented by correspondences and publications, and that the parties were clearly aware of the controversial character of their discussions. Consequently, we can consider historical facts and avoid speculations that would not be supported by our sources. This, at least, is what I hope to be able to do.

It seems dangerous to draw conclusions with some relevance to the contributions of the philosophers from a single case study. So I will instead discuss (in sections II through VII) a series of controversies all of which are closely related to infinity in the real and complex analysis of the 18th and 19th centuries: the development of knowledge about a single function, the logarithm, from Leibniz to Riemann; the debate about “general functions” in analysis and physics, beginning around the middle of the 18th century; Cauchy’s attempt to reconcile rigor with the intuition of infinitely small quantities; the dialectic of the continuous and the discrete in the foundational work of Riemann and Dedekind; and the debate about transfinite numbers near the end of the 19th century. These case studies will serve as material for responses to the papers by Breger, Cellucci and Posy in sections VIII through XII.

Herbert Breger considers the development of mathematics from the perspective of levels of abstraction. This turns out to be methodologically fruitful. Indeed, numbers and functions, and problems associated with them, reappeared at different stages, and were reformulated and treated repeatedly in connection with attempts to resolve some controversies. Breger and Cellucci discuss the problem of “tacit knowledge.” It turns out that controversies can give us information about the tacit knowledge of a scientific community, or rather, they may indicate that, at some point, the parties no longer agree.
about earlier hidden assumptions.

Like Breger, I avoid the term “conceptual framework.” As Carlo Cellucci points out, relevant growth of knowledge normally goes with the analytic method rather than with some rigid framework supplied in the guise of the synthetic method. The latter method, however, is vital for the consolidation and transmission of knowledge in the form of textbooks. These are usually overestimated by historians, while preceding debates provide insight into the process of reformulation and resolution of problems.

Infinity in analysis is closely related to the continuity of functions and to the linear continuum of “numbers.” I had not intended to discuss these here, since I felt biased by my own involvement in the invention of nonstandard analysis. But when the organizers decided shortly before the present conference that I was to respond to Carl Posy’s paper on epistemology, ontology and the continuum, I readily agreed to do so. The discussion encouraged me to apply some ideas I found in Cauchy’s work to questions raised by Posy in his exposition and criticism of Brouwer. This is another example of how one and the same problem can be considered on different levels of conceptual abstraction.

THE CONTROVERSY OVER THE LOGARITHMS
OF NEGATIVE AND IMAGINARY (COMPLEX) NUMBERS (1712-1761)

In a paper of 1712, and in his correspondence with Johann Bernoulli during the years 1712-13, Leibniz argued that the logarithms of negative numbers could not be real, while Bernoulli sought to prove \( \log(-a) = \log a \). Each gave several reasons for his opinion. Leibniz’s main objection against the reality of \( \log(-1) \) was that the value of an exponential function was positive for any real exponent, as could also be seen from the exponential series. The strongest of Bernoulli’s reasons was the functional equation \( \log(ab) = \log a + \log b \), from which he deduced \( 0 = \log 1 = \log(-1)^2 = 2 \log(-1) \).

In his early correspondence with Bernoulli (1727-31), Euler disagreed with the opinion of his teacher, though at the time he had no clear position of his own. Leibniz’s and Johann Bernoulli’s correspondence appeared in print in 1745, and at that point, Euler could settle the controversy. Indeed his formula (with \( i \) for Euler’s \( \sqrt{-1} \))

\[
z = e^{i(\alpha+2\pi n)} = \cos(\alpha + 2\pi n) + i \cdot \sin(\alpha + 2\pi n) = \cos \alpha + i \cdot \sin \alpha = e^{i\alpha}
\]

immediately yielded \( \log z = i(\alpha+2\pi n) \) for all integers \( n \). Obviously, each logarithm had infinitely many values.

This did not convince his contemporaries, including d’Alembert, who would stick to Bernoulli’s opinion until 1761. So Euler decided to tackle the matter by a more direct approach, in the algebraic style of his Introductio in analysin infinitorum (written in 1745, published in 1748). In his long paper of 1749, Euler started from his representation of the exponential function \( x = e^y \) by

\[
x = (1 + \frac{y}{n})^n
\]

“posant le nombre n infiniment grand.” Then,

\[
\log x = y = nx^{1/n} - n
\]