Philosophers frequently link their discussions of progress in science and mathematics to the issues of scientific and mathematical realism. I don’t dispute that these connections can be made, but I think that questions of progress in mathematics and science are more complicated than this, and that perhaps the more important measures of progress are independent of questions of realism. So I want to begin by distinguishing several senses in which we might measure progress in mathematics. My investigation on this front ends on a pessimistic note: perhaps we can establish that mathematics as a whole makes progress, but it is unlikely that we can measure progress in one branch of mathematics or in one historical period against that in another.

In my mind the issue of mathematical progress is more closely connected to questions of value realism than to questions of mathematical realism. Still I don’t think we should neglect Philip Kitcher’s objection that mathematical realists cannot satisfactorily account for mathematical progress, and I turn to it later in the paper. As it turns out this objection also raises a very fundamental question about the epistemology of mathematics. In a final section, I try to show that my own structuralist version of realism can respond to this question and Kitcher’s objection.

RELATIVE PROGRESS AND PROGRESS RELATIVE TO GOALS

To assess something’s progress is to evaluate it, for in making progress we do better, in retrograding we do worse. However, when it comes to progress, there are different ways of doing better. One way is to be moving towards specific, articulated goals that we acknowledge and endorse — as when one solves an open mathematical problem. Another way is simply to be making a positive shift along a dimension that we value. For example, we might praise a contemporary branch of mathematics for formulating more abstract and general versions of old results.

Obviously, we change our goals when we attain them or find them to be infeasible or ill defined. Graph theorists, I take it, no longer aim to decide the Four Color Conjecture, although they may well want to give a surveyable proof of it. In the wake of Gödel’s underivability theorems, to give another example, a whole new set of goals took the place of Hilbert’s goal of giving a finitary proof of the consistency of number theory.

Furthermore, the progress we make towards a goal can be assessed along different dimensions. For example, during the past fifty years we have amassed a considerable body of results that bears upon the continuum problem. This can be said to represent significant progress towards resolving it, and it is easy to argue that the
progress we have made on the continuum problem during the past fifty years far outstrips that made during the first fifty years following Cantor's introduction of it. On the other hand, it can also be argued that while we know much more about the continuum hypothesis than we did one hundred years ago, we are no closer to deciding its truth, and so have not made any "real" progress towards our original goal.

The same obviously holds true when we assess progress without referring to goals. For example, to me some of the contemporary text book presentations of Gödel's, Church's and Tarski's theorems are superior, in a sense, to those I learned as a graduate student because they are simpler and based upon one general diagonalization lemma. But, in another sense, they are too condensed, suppress some of the information contained in the earlier proofs, and omit the intuitions historically motivating those proofs.

So far I have distinguished progress that is oriented towards goals from that which is not. But to simplify my exposition I am going to conflate these two notions by assuming that each branch of mathematics also has the vague goal of "doing better" and that its progress towards this goal is assessed along whatever dimensions it uses to mark improvements that do not refer to explicit goals. Suppose, for example, that no one has articulated the goal of simplifying the proof of the consistency of the continuum hypothesis, but that doing so would still count as improving set theory. Then I will take finding such a proof as a way of making progress along one dimension of the vague goal of "doing better" in set theory. This will allow us to represent all types of mathematical progress as progress towards some goal G measured along some dimension D.

Let us suppose that we can score the progress that a given branch of mathematics, set theory, for instance, has made towards one of its goals, say, of doing better, as measured along a given dimension, say, of obtaining simpler proofs. We still may not be in a position to determine whether this branch has made progress towards that goal. For it may have made some progress along some dimensions while stagnating or retrograding along others. Unless priorities or weights exist for amalgamating scores along the various dimensions there may not even be a fact of the matter as to whether progress has been made towards the goal in question. What is more, by the same token, even if there are well-defined notions of progress towards each of the goals motivating a given branch of mathematics, there may still be no fact to the matter as to whether the branch as a whole has made progress. For once again progress toward one goal may not be comparable with that towards another.

These considerations also apply to the problem of assessing progress in mathematics as whole, where the problems are compounded by the fact that mathematics has many branches with different aims and open problems. Furthermore, it may well be that progress in one branch of mathematics spills over into other branches of mathematics, just as advances in algebra led to advances in mathematical logic. Probably all branches of mathematics share some ways of marking improvements. It is likely that they all prefer simpler, more elegant, more general, and more abstract treatments of any given body of results. But again this will not help if we must balance gains in simplicity in geometry against a loss of rigor in analysis.