Chapter 4
Monads in General Topology

The set-theoretic stance of mathematics has provided us with the environment known today as general topology for studying continuity and proximity since the beginning of the 20th century.

Considering the microstructure of the real axis, we have already seen that the collection of infinitesimals arises within infinitesimal analysis as a monad, i.e., the external intersection of all standard elements of the neighborhood filter of zero in the only separated topology agreeable with the algebraic structure of the field of reals.

We may say that the notion of the monad of a filter synthesizes to some extent the topological idea of proximity and the analytical idea of infinitesimality. Interplay between these ideas is the main topic of the current chapter.

We focus attention on the most elaborate ways of studying classical topological concepts and constructions that surround compactness and rest on the idealization principle we accept in nonstandard set theory.

The contribution of the new approach to the topic we discuss resides basically in evoking the crucial notion of a nearstandard point. The corresponding test for a standard space to be compact, consisting in nearstandardness of every point, demonstrates the meaning and significance of the concept of nearstandardness which translates the conventional notion of compactness from whole spaces to individual points. This technique of individualization is a powerful and serviceable weapon in the toolbox of infinitesimal analysis.

It is worth observing that we conform mainly to the neoclassical credo of infinitesimal analysis in this chapter, and so exposition proceeds in the standard environment unless otherwise stated.

4.1. Monads and Filters

The simplest example of a filter is well-known to be the collection of supersets of
a nonempty set. Infinitesimal analysis enables us to approach an arbitrary standard filter in much the same manner viewing this filter as standardization of the collection of supersets of an appropriate external set, the monad of the filter. The method for introducing these monads and studying their simplest properties constitutes the topic of this section.

4.1.1. Let \( X \) be a standard set and let \( \mathcal{B} \) be a standard filterbase on \( X \). In particular, \( \mathcal{B} \neq \emptyset \), \( \mathcal{B} \subset \mathcal{P}(X) \), \( \emptyset \notin \mathcal{B} \), and \( B_1, B_2 \in \mathcal{B} \rightarrow (\exists B \in \mathcal{B})(B \subset B_1 \cap B_2) \). The symbol \( \mu(\mathcal{B}) \) denotes the monad of \( \mathcal{B} \), i.e., the external set defined as follows:

\[
\mu(\mathcal{B}) := \bigcap\{B : B \in \text{^0}\mathcal{B}\}.
\]

4.1.2. An internal set \( A \) is a superset of some standard element of a standard filterbase \( \mathcal{B} \) if and only if \( A \) includes the monad \( \mu(\mathcal{B}) \) of \( \mathcal{B} \).

\(<\) If \( A \supset B \) and \( B \in \text{^0}\mathcal{B} \) then \( A \supset \mu(\mathcal{B}) \) by definition. Conversely, if \( A \supset \mu(\mathcal{B}) \) then by idealization there is an internal set \( B \in \mathcal{B} \) such that \( B \subset \mu(\mathcal{B}) \), and so \( A \supset B \). \(\triangleright\)

4.1.3. Each standard filter \( \mathcal{F} \) is the standardization of the principal external filter of supersets of the monad \( \mu(\mathcal{F}) \).

\(<\) In symbols, the claim reads:

\[
(\forall^{\text{st}} A)((A \in \mathcal{F}) \leftrightarrow (A \supset \mu(\mathcal{F}))).
\]

The last equivalence is obviously a consequence of 4.1.2. \(\triangleright\)

4.1.4. The monad \( \mu(\mathcal{F}) \) of a standard filter \( \mathcal{F} \) is an internal set if and only if \( \mu(\mathcal{F}) \) is a standard set. In this event \( \mathcal{F} \) is the standard filter of supersets of \( \mu(\mathcal{F}) \).

\(<\) If \( \mu(\mathcal{F}) \) is an internal set then, using 4.1.3 and the idealization principle, we find

\[
(\exists A)(\forall^{\text{st}} F)(F \in \mathcal{F} \leftrightarrow (F \supset A) \leftrightarrow (\forall^{\text{fin}} \mathcal{U})(\exists A)(\forall F \in \mathcal{U})(F \in \mathcal{F} \leftrightarrow F \supset A)) \\
\leftrightarrow (\forall^{\text{st}} \mathcal{U})(\exists A)(U \in \mathcal{F} \leftrightarrow U \supset A).
\]

By transfer, we conclude that \( \mathcal{F} \) is the filter of supersets of some set \( A \). Since such a set \( A \) is unique; therefore, \( A = \mu(\mathcal{F}) \) and, moreover, \( A \) is a standard set. \(\triangleright\)

4.1.5. Given a standard filterbase \( \mathcal{B} \), we call the members of \( \mu(\mathcal{B}) \) infinitesimal or distant, or remote, or astray (relative to \( \mathcal{B} \)). Analogously, an element \( B \) in \( \mathcal{B} \) such that \( B \subset \mu(\mathcal{B}) \) is also called an infinitesimal or distant, or remote, or astray member of \( \mathcal{B} \). The collection of infinitesimal members of \( \mathcal{B} \) is denoted by \( ^a\mathcal{B} \).