In addition to first-order closure models reviewed in the previous chapter, second-order closure schemes are discussed in the present one to complete the review on single point modeling. Accordingly, this chapter is intended as providing some insights on where second-order turbulence models have reached in accounting for several distinct effects due to density variation in low speed motions and compressibility in high Mach number flows.

11.1. Introduction

In incompressible fluid motion, second-order turbulence modeling is concerned with the closure of transport equations for second-order moments: \( \rho_0' u'_i u'_j \), \( \rho_0^2' f' \), and \( \rho_0 f'^2 \), where \( f' \) stands for any turbulent fluctuation of a passive scalar contaminant and \( \rho_0 \) is the (constant) density of the fluid.

In variable density fluid motions, a similar approach can be adopted, based on equivalent correlations which are now including density variations, such as \( \rho u'_i u_i' \), with a ternary regrouping (centered fluctuations), or \( \rho u_i'' u_i'' \), with a binary regrouping using Favrian fluctuations (see Chapter 5).

From a general point of view, second-order modeling in variable density turbulent flows addresses transport equations for the following moments \( \rho u_i'' u_j'' \), \( \rho \theta'' u_j'' \), \( \rho \gamma'' u_j'' \), \( \rho \theta'' \gamma'' \) and \( \rho \gamma'' \gamma'' \), where \( \theta'' \) and \( \gamma'' \) denote the Favrian fluctuations of temperature \( (T) \) and mass fraction \( (C) \) in a binary mixture respectively.

To avoid solving an increasing number of additional transport equations in compressible flows, scalar correlations are often derived from a generalized gradient diffusion assumption. In this case, second-order modeling is mostly concerned with deriving closure schemes for the Reynolds stress transport equation. This question will be thus discussed first.
11.2. **The modeling issue of RST equation**

11.2.1. **THE OPEN TRANSPORT EQUATION**

The Reynolds stress transport equation in a compressible fluid motion has been derived in Chapter 5. By simple rearrangement of the right-hand-side terms of eq.(5.36) and using Favre's averaging ($\widetilde{u'} u'' \\ j \equiv \rho u'' u'' / \bar{\rho}$), it reads (see also Chapter 9):

$$
\frac{\partial (\bar{\rho} u'' u''_j)}{\partial t} + \frac{\partial (\bar{\rho} u'' u''_j \bar{U}_k)}{\partial x_k} = \bar{\rho} P_{ij} - \frac{\partial (T_{ijk})}{\partial x_k} + \bar{\rho} \Pi^*_{ij} + \Sigma_{ij} - \bar{\varepsilon}_{ij}, \quad (11.1)
$$

where the right-hand-side contributions are:

- **Turbulent production:** $P_{ij} = -\left( \bar{u''}_i \bar{u''}_j \frac{\partial \bar{U}_k}{\partial x_k} + \bar{u''}_j \bar{u''}_k \frac{\partial \bar{U}_i}{\partial x_k} \right)$,

- **Pressure strain:** $\Pi^*_{ij} = \frac{1}{\bar{\rho}} \left[ p' \left( \frac{\partial \bar{u''}_i}{\partial x_j} + \frac{\partial \bar{u''}_j}{\partial x_i} \right) \right]$,

- **Transport:** $T_{ijk} = \bar{\rho} u''_i \bar{u''}_j \bar{u''}_k + p' u' \delta_{jk} + p' u'_j \delta_{ik} - (\tau_{jk} \bar{u'}_i + \tau_{ik} \bar{u'}_j)$,

- **Mass flux coupling:** $\Sigma_{ij} = \bar{u''}_i \left( \frac{\partial \tau_{jk}}{\partial x_k} - \frac{\partial \bar{P}}{\partial x_j} \right) + \bar{u''}_j \left( \frac{\partial \tau_{ik}}{\partial x_k} - \frac{\partial \bar{P}}{\partial x_i} \right)$,

- **Turbulent dissipation:** $\bar{\varepsilon}_{ij} = \left( \tau_{jk} \frac{\partial \bar{u''}_i}{\partial x_k} + \tau_{ik} \frac{\partial \bar{u''}_j}{\partial x_k} \right)$.

Two formal modifications can be introduced in eq.(11.1), as inferred from the analysis of the constant density situation.

a) In isovolume turbulent fluid motions, the pressure-strain term is traceless ($\Pi^*_{ii} (I) \equiv 0$), which means that it is only responsible for a redistribution of energy between the normal stresses, without changing the total amount of turbulence kinetic energy. The same characteristic can be recovered in variable density fluid motions, when considering the deviatoric part of the pressure-strain correlation:

$$
\bar{\rho} \Pi_{ij} = \bar{\rho} \Pi^*_{ij} - \frac{2}{3} \bar{P} \bar{\vartheta} \delta_{ij}, \quad (11.2)
$$

where $\bar{\vartheta} = \partial u'_i / \partial x_i$. It is recalled that $u'_i = u''_i - \bar{u''}_i$ (see Tab.5.2), so that $\bar{P} \bar{\vartheta} \equiv \bar{p'} \bar{\vartheta}^{\prime\prime}$. 