CHAPTER 9
NONLINEAR ANALYSIS OF SHELLS

9.1 Introduction

This chapter extends the material of Chapter 6, Nonlinear Analysis of Membranes, to include the effects of bending and the material of Chapter 5, Nonlinear Analysis of Space Frames, to remove rigid body rotations (Levy and Gal 2002). Chapter 6 discussed a “membrane shell”, using a finite element developed for fabric structures applications where the stress levels are almost constant. The approach to shell buckling of this chapter is quite general, building upon this work in Chapter 6.

The history of finite element applications to shell buckling is extensive going back to the work of Clough and Johnson (1968). The natural mode contribution of Argyris et al. (1977) was a major addition to shell theory. It was recently modified to include elastoplastic effects (Argyris et al. 2000). Horrigmoe and Bergan (1978) used the co-rotational method for nonlinear analysis and Bathe and Ho (1980), Hsiao (1987), Mohan and Kapania (1997), Peng and Crisfield (1992) improved element performance along those lines. The 3-D elasticity "degenerate" element of Ahmad et al. (1970) was followed by, among others, Bathe and Balourchi (1980), Hughes and Lui (1981), Dvorkin and Bathe (1984), and Buechter and Ramm (1992). In an excellent review, Ibrahimbegović (1997) addresses the various approaches and the complex issues involved.

Here the derivation of the geometric stiffness matrix is somewhat different but consistent with the approach used throughout this text. The linear equilibrium equations for a flat triangular shell element in its local coordinates system are first perturbed to yield the in-plane geometric stiffness matrix. Then out-of-plane considerations that involve the effect of rigid body rotations on member forces yield an out-of-plane geometric stiffness matrix. The shell element that was chosen for that purpose combines the constant stress triangle (CST) flat triangular membrane element (Zienkeiwicz (1977)) and of the discrete Kirchhoff theory (DKT) flat triangular plate element (Batoz et al. (1980)).

Finally a computer program, featuring incremental analysis and Newton’s method, geometric effects, pure deformation isolation, internal stress retrieval and updating of nodal forces and coordinates is presented and used to solve a number of problems that have appeared in the literature.
These problems include experiments, analytical results and numerical problems that were solved by the finite element method. They compare well with the results in the current literature. Some details in the derivation have been omitted. The interested reader can find them in Gal (2002).

9.2. The Geometric Stiffness Matrix of Triangular Element Shells

The local geometric stiffness matrix of the shell element is split into three distinct matrices:

\[
\begin{bmatrix} K^e_{G} \end{bmatrix}_{\text{TOTAL}} = \begin{bmatrix} K^e_{G} \end{bmatrix}_{\text{mem}} + \begin{bmatrix} K^e_{G} \end{bmatrix}_{\text{plate}} + \begin{bmatrix} K^e_{G} \end{bmatrix}_{\text{OP}} \tag{9.1}
\]

where the first, second and third terms on the R.H.S. of Eq. 9.1 represent the in-plane geometric stiffness matrix of the membrane, the in-plane geometric stiffness matrix of the plate and the out-of-plane geometric stiffness matrix of the shell element respectively. The total, ‘tangential’ stiffness matrix for use in nonlinear analysis will include, in addition, the linear elastic stiffness matrices of a plane stress triangular element (membrane) and that of a triangular plate element.

The geometrically nonlinear triangular shell element has eighteen local degrees of freedom (DOF's): 3 displacements and 3 rotations at each node. The membrane element contributes to nine displacement DOF's only. The basic three noded constant stress triangular flat element has only six local displacement DOF's that are shown in Figure 9.1. The out-of-plane contribution (the normal stiffness) of the membrane element to the basic local shell element is a displacement DOF in the direction normal to the plane of the element. This nonzero contribution was encountered earlier in prestressed trusses as the ability to carry load perpendicular to the bars.

![Figure 9.1 Triangular membrane element.](image-url)