Chapter 3. THE HOMEOTOPY GROUP OF A SURFACE

3.1. The homeotopy group

In the topology of three-dimensional manifolds, we often have to glue manifolds together by means of various different homeomorphisms of their boundaries. Gluing by isotopic homeomorphisms gives one and the same result (this is proved below). It is therefore reasonable to investigate the group of homeomorphisms of a surface onto itself modulo homeomorphisms isotopic to the identity. Let $F$ be a surface (perhaps, with boundary). The homeotopy group $H(F)$ of the surface $F$ is defined as a quotient group of the group of homeomorphisms of the surface $F$ onto itself with respect to the subgroup $\text{Iso}(F)$ of homeomorphisms isotopic to the identity. If $h : F \to F$, $h_t = h$, $h_0 = 1$ is an isotopy of the homeomorphism $h$ to the identity, then $fh_t f^{-1}$ is an isotopy of the conjugate homeomorphism $fhf^{-1}$ to the identity. Therefore, the subgroup $\text{Iso}(F)$ is normal. It is usually more convenient to consider the group of homeotopies $H(F, \partial F)$ fixed on the boundary. Each of its elements is defined by a homeomorphism fixed on the boundary, two homeomorphisms determining one and the same element if and only if they are isotopic under an isotopy fixed on the boundary.

3.2. Twists

A simple example of a non-trivial self-homeomorphism of a surface is a twist along a curve. Let $c$ be a simple closed curve in an orientable surface $F$. Cut the surface $F$ along the curve $c$, twist one of the edges of the cut by $360^\circ$ in one of the two possible directions and glue the edges of the cut back together, as shown in Fig. 74. The homeomorphism $\tau_c : F \to F$ thus obtained is called a twist along the curve $c$. It can be taken fixed outside of $c$. 

![Figure 74](image-url)
an annulus \( U(c) \subset F \), one of the components of whose boundary is the curve \( c \). If the annulus \( U(c) \) is identified with the annulus \( \{ z : 1 \leq |z| \leq 2 \} \) in the complex plane, the twist \( \tau_c \) can be given by the rule \( re^{i\theta} \rightarrow re^{i(\theta + 2\pi(r-1))} \) on the annulus \( U(c) \) and by the identity outside this annulus. The choice of another annulus \( U(c'), \partial U(c') \supseteq c \), as well as the replacement of the curve \( c \) by an isotopic one, leads to isotopic twists. Essential is only the choice of the twist direction. Here, a twist in the opposite direction determines the inverse element of the homeotopy group.

Twists, which are really quite simple homeomorphisms, play an important role in the study of homeotopy groups. It turns out that the group \( H(F, \partial F) \) is generated by a finite number of twists. In other words, in any compact orientable surface \( F \) there exists a finite set of simple closed curves \( c_1, c_2, \ldots, c_n \) such that the twists \( \tau_{c_i} \) generate the group \( H(F, \partial F) \).

The scheme of the proof of this result is as follows. First we prove it for a disc, then for a