1. Set-valued mappings

Here we deal with some properties of set-valued mappings. These properties will be applied in our further considerations. The notion of a set-valued mapping is a generalization (in a certain sense) of the notion of an ordinary mapping. First, let us recall that a set-valued mapping (or a multi-valued mapping, or a multi-valued function) is a mapping of the type

$$F : X \rightarrow P(Y)$$

where $X$ and $Y$ are arbitrary sets and the symbol $P(Y)$ denotes the family of all subsets of $Y$. In other words, for each element $x \in X$, we have $F(x) \subseteq Y$.

It is easy to see that the notion of a set-valued mapping is equivalent, in some sense, to the notion of a binary relation between two sets. Indeed, let $X$ and $Y$ be any two sets and let $X \times Y$ be the Cartesian product of these sets. Furthermore, let $G$ be a binary relation between $X$ and $Y$, i.e. $G \subseteq X \times Y$. Then this relation canonically defines a set-valued mapping

$$F_G : X \rightarrow P(Y)$$

given by the formula

$$F_G(x) = G(x) = \{y \in Y : (x, y) \in G\}$$

where $x$ is an arbitrary element of $X$. As usual, $F_G$ is called the set-valued mapping canonically associated with the original binary relation $G$.

Conversely, if we have a set-valued mapping $F : X \rightarrow P(Y)$, then we can define a binary relation $G_F$ between the sets $X$ and $Y$ by the following formula:

$$(x, y) \in G_F \iff y \in F(x).$$

As usual, $G_F$ is called the binary relation canonically associated with the original set-valued mapping $F$. Notice that $G_F$ is also called the graph of the set-valued mapping $F$.

If we consider a set-valued mapping $\Phi$ canonically associated with the binary relation $G_F$, then it is easy to check that $\Phi$ coincides with the original set-valued mapping $F$. 

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We also wish to remark that, in many cases, it is possible to restrict considerations only to those set-valued mappings $F$ for which the condition

$$(\forall x \in X)(F(x) \neq \emptyset)$$

is fulfilled. Finally, notice that any ordinary mapping $f : X \rightarrow Y$ canonically defines a set-valued mapping $F_f : X \rightarrow P(Y)$ given by the formula

$$F_f(x) = \{f(x)\} \quad (x \in X).$$

Thus, as mentioned above, ordinary mappings are a very particular case of set-valued mappings. Of course, we can associate several set-valued mappings with an ordinary mapping $f : X \rightarrow Y$. For instance, we can define a set-valued mapping $F : Y \rightarrow P(X)$ by the following formula:

$$F(y) = f^{-1}(y) \quad (y \in Y).$$

There are many interesting and important facts concerning set-valued mappings. For example, the well-known theorem of Cantor, stating that

$$\kappa < 2^\kappa$$

for all cardinal numbers $\kappa$, can be formulated in terms of set-valued mappings. Actually, this theorem says that, for every set $X$ and for any set-valued mapping of the type

$$F : X \rightarrow P(X),$$

the relation $\text{ran}(F) \neq P(X)$ is true (i.e. $F$ is not a surjection). Indeed, let

$$G_F = \{(x,y) \in X \times X : y \in F(x)\}$$

denote the graph of $F$. Consider the set $Z = \{(x,x) : x \in X\} \setminus G_F$ and put $D = \text{pr}_1(Z)$. Suppose for a moment that $D \in \text{ran}(F)$. Then there exists an element $x_0 \in X$ such that $D = F(x_0)$. Now, it is easy to check the equivalence

$$x_0 \in D \iff x_0 \notin D$$

which obviously yields a contradiction. We thus conclude that $D \notin \text{ran}(F)$ and $\text{ran}(F) \neq P(X)$.

Some other facts and assertions concerning set-valued mappings will be discussed below.

We begin with the discussion of one statement equivalent to the negation of the Continuum Hypothesis. This statement is due to Freiling (see