9. MODIFICATIONS TO THE HIERARCHICAL NORMAL LINEAR MODEL

All the modifications discussed in this Chapter have to do with relaxing the assumption of normality. The first two cover specific distribution choices—lognormal and Poisson. In all cases the normal distribution is retained for the second level. This can usually be accomplished by careful parametrization of the first level parameters. The third modification presented is a general method for dealing with non-normal distributions.

A. LOGNORMAL

This is an especially easy case due to the simple relationship between the normal and lognormal distributions. Assume that at the first level each observation has a lognormal distribution. That is,

\[ \ln(X_{ij}) \sim N(\theta_i, \sigma^2/P_{ij}) \]  

(9.1)

where \( X_{ij} \) is losses per exposure. All the customary formulas can be used once natural logarithms are taken of all of the observations. The major drawback to this model is that the mean (of \( X_{ij} \)) depends on the exposure. In particular, for predicting future observations we have to compute the following posterior expectation:

\[ E(e^\theta + \sigma^2/2P | x). \]  

(9.2)

Again we see that the value depends on the exposure.
One way to alleviate this problem is to change the first level model to

\[
\ln(X_{ij}) \sim N(\theta_i - \sigma^2/2P_{ij}, \sigma^2/P_{ij})
\]  

(9.3)

or, equivalently,

\[
\ln(X_{ij}) + \sigma^2/2P_{ij} \sim N(\theta_i, \sigma^2/P_{ij})
\]  

(9.4)

The expected value of \(X_{ij}\) is now simply \(\exp(\theta_{ij})\). The problem now is that \(\sigma^2\) must be known (or estimated) in advance.

B. POISSON

The appropriate method for handling the Poisson distribution was introduced in Chapter 6. The key is a variance stabilizing transformation. We make the additional assumption that this produces values that approximately normal. As seen in Chapter 6, the skewness is close to zero, so this is not an unreasonable statement. The use of this transformation in the HNLM is illustrated with Data Set 2 and an analysis that parallels the presentation in the previous Chapter.

Recall that \(Y_{ij}\) is the frequency count and it is reasonable to assume that it has the Poisson distribution with parameter \(\alpha P_{ij}\lambda_i\) where \(\alpha P_{ij}\) is the number of opportunities to have an accident (the constant factor \(\alpha\) is an adjustment to reflect the fact that \(P_{ij}\) is only proportional to this number), \(\lambda_i\) is the expected number of claims per opportunity for members of the \(i\)th class. The first level observation is \(x_{ij} = Y_{ij}/P_{ij}\) and the first level parameter is \(\theta_i = E(x_{ij} | \alpha, \lambda_i) = \alpha \lambda_i\). The transformation is \(z_{ij} = 2\sqrt{x_{ij}}\) and its mean and variance are approximately \(\gamma_i = 2\sqrt{\theta_i}\) and \(1/P_{ij}\) respectively. The one-way model can now be used with \(\gamma_i\) replacing \(\theta_i\) and \(\sigma^2 = 1\). This last replacement is especially convenient as all integrals will be one-dimensional.

Using the prior \(\pi_{22}(\tau^2) \propto 1/\tau^2\) and the density in (7.48) along with adaptive Gaussian integration yields \(E(\tau^2 | z) = 0.02951\). The empirical Bayes estimate of \(\gamma_i\) is

\[
\tilde{\gamma}_i = \frac{P_i}{P_i + 33.891} \tilde{\gamma}_i + \frac{33.891}{P_i + 33.891} \hat{\mu},
\]