RAYMOND M. SMULLYAN

CHAMELEONIC LANGUAGES

INTRODUCTION

We would like to open with a report of a delightful remark from a correspondent who wrote: "I have heard of your chameleonic languages. I do not know what they are, except that I assume they are not what they appear to be."

Now, there are certain words in the English language whose denotation is dependent on the surrounding context, and which might accordingly be labeled *chameleonic*. (*Indexical* is the more usual term, but the more colorful word *chameleonic* seems a bit more appropriate for this article.) For example, the word *now* denotes different instants of time when uttered at different times; the word *I* designates different people when used by different persons[1]; the word *you* designates the person to whom it is addressed. Like a chameleon whose color depends on its surroundings, these words change their denotation from context to context. A classical example of a chameleonic term, and one which appears in the well-known formulation of the liar paradox, is the word *this* – in such contexts as "This sentence has property P." Such a sentence might be termed *self-referential*, in that it expresses the proposition that it itself has the property P, and is accordingly true if and only if it does have the property P. Now, the usual first-order arithmetic theories, in which one is interested in obtaining incompleteness and undecidability results, do not contain chameleonic symbols. (One gets around this by the use of Gödel's diagonal function [2], or for some languages, the simpler norm function introduced in [3]). In this article, instead of circumventing the chameleonic term, we formalize it directly.

In Part 1 we work in informal English. Given a property P of sentences, we show how to construct a nonindexical sentence (a *normal* sentence, as we call it) which says that it has property P. Then we consider the problem of *cross-reference* – given two properties P_1, P_2, how to construct normal sentences S_1, S_2, such that S_1 says that S_2 has property P_2, and S_2 says that S_1 has property P_1. (This is related to the

Synthese **60** (1984) 201–224. 0039–7857/84/0602–0201 $02.40

doubly Gödelian islands of [4].) Then we consider the problem of *simultaneous self-reference* – given two binary relations $R_1(X, Y)$, $R_2(X, Y)$ on sentences, how to construct normal sentences S_1, S_2, such that S_1 says that $R_1(S_1, S_2)$, and S_2 says that $R_2(S_1, S_2)$. (This is related to the author's double-recursion theorem [5], or to the closely related generalized diagonal lemma of Boolos [1].) We have found two quite different methods of achieving cross-reference and simultaneous self-reference: the first uses two indexical terms "this" and "that"; the second uses only the one indexical term "this". The first method when formalized in arithmetic yields essentially the construction of Boolos [1]; the second yields results which are more novel (Theorems 5, 6, 7, 8, 9 in Part 2).

In Part 2 we arithmetize some of the constructions of Part 1. The gist of the idea is that we take a first-order arithmetic theory (T), add a new individual symbol "σ" (the *chameleonic* symbol) and in any sentence C in which "σ" occurs (a "chameleonic" sentence, as we call it), the symbol "σ" is interpreted as denoting the Gödel number of the entire sentence C. (Thus these chameleonic sentences refer to their own Gödel numbers in as direct a manner as can be imagined.) Each chameleonic sentence C is assigned an ordinary sentence called the *translation* of C, which is true if and only if C is chameleonically true (true, that is, when σ is interpreted as being the Gödel number of C). Given a subset P of the set of true nonchameleonic sentences (whose elements are the *provable* sentences, say in Peano Arithmetic), we let P^σ be the set of all chameleonic sentences whose translation is in P. Both the set T^σ (the set of chameleonically true sentences) and the set P^σ have bizarre properties; they are neither consistent nor closed under modus ponens. (Though they are inconsistent, no single member is inconsistent.) Despite their bizarre properties, these sets are far from useless – for example, one way to get an undecidable sentence of the original theory is to represent in it the set of Gödel numbers of the set P^σ. This method becomes virtually Gödel's diagonal method if we replace "σ" by a variable. It is no simpler than the standard method, but is heuristically illuminating.

After carrying out various "multiply self-referential" constructions in arithmetic theories, we concluded with a consideration of the following problem: A chameleonic formula $K(x,\sigma)$ (which has x as the only free variable, but also may have the chameleonic symbol σ) can be said to represent a set – namely the set of all n such that the chameleonic