CHAPTER I : FUNCTIONAL INTEGRALS DEFINED AS LIMITS OF
DISCRETIZED EXPRESSIONS

1.1. Introduction

The functional solution or propagator $K(q,t;q_0,t_0)$ of a first order
(in time $t$) partial differential equation

$$[\frac{\partial}{\partial t} - L(q,\dot{q},t)] \phi(q,t) = 0$$

is defined as its solution satisfying the initial condition

$$K(q,t_0;q_0,t_0) = \delta(q - q_0)$$

Boundary conditions in $q$-space can be specified and $L(q,\dot{q},t)$ is a linear,
not necessarily hermitean and possibly time dependent operator. ($\partial \equiv \frac{\partial}{\partial q}$)
In the following we shall often restrict it to be of second order in $\dot{q}$.
It can then always be written in the form

$$L(q,\dot{q},t) = \frac{1}{2} c \sum_{\mu,\nu} \frac{\partial}{\partial q^\mu} G(q,t) + \sum_{\mu} A^\mu(q,t) + \frac{1}{c} V(q,t)$$

where the range of the Greek indices is from 1 to $d$, $d$ being the dimension
of $q$-space. The number $c$ in (1.1) is in general complex with positive real
part. For $c = i$ eq. (1.1) is a Schrödinger equation, for $c = 1$ it is a
Fokker-Planck or diffusion type equation. The fundamental property of $K$
leading to its path integral representation is the semigroup law

$$K(q,t;q_0,t_0) = \int dq' K(q,t;q',t') K(q',t';q_0,t_0)$$

In Fokker-Planck dynamics, where $K$ has the interpretation of a transition
probability density, this is called the Chapman-Kolmogorov equation. It
expresses the Markov property of the stochastic process $q$ (see Chapter 7).

A functional integral representation of $K$ can be obtained in the
following way. Divide the time interval $[t_0,t]$ into $N + 1$ pieces of length
$\varepsilon = (t - t_0)/(N+1)$, denote the intermediate times by
\[ t_j = t_0 + j \varepsilon, \quad j = 0, 1, \ldots, N+1 \]

\[ t_{N+1} = t \]

and iterate (1.4). One obtains \((q_{N+1} \equiv q)\)

\[
K(q, t; q_0, t_0) = \int dq_N dq_{N-1} \ldots dq_1 \ K(q, t; q_N, t_N) \ K(q_N, t_N; q_{N-1}, t_{N-1}) \\
\ldots \ K(q_1, t_1; q_0, t_0)
\]

\[
= \prod_{i=1}^{N} dq_i \prod_{j=1}^{N+1} K(q_j, t_j; q_{j-1}, t_{j-1})
\]

(1.6)

We are interested in the limit of (1.6) for \(N \to \infty\). In this limit \(e = t_j - t_{j-1}\) becomes arbitrarily small and \(K(q_j, t_j; q_{j-1}, t_{j-1})\) is called the short time propagator. The useful aspect of the procedure is that (if the limit exists) every short time propagator contributes to the final result \(K(q, t; q_0, t_0)\) with terms up to order \(e\) only. Once this approximation is determined, a functional integral representation can be defined.

It was assumed by Feynman \([F 3]\), following Dirac \([D 19]\) that in quantum mechanics it is proportional to \(\exp \left[ \frac{1}{\hbar} A(q_j, t_j; q_{j-1}, t_{j-1}) \right] \) with \(A\) the classical action of the classical path between \((q_j, t_j)\) and \((q_{j-1}, t_{j-1})\).

Also approximations of this function were used. \([F 3, F 5]\)

Let us consider the example of the Schrödinger equation for a particle in a one dimensional harmonic oscillator potential:

\[
\frac{i}{\hbar} \frac{3}{3t} \psi(q, t) = \left[ -\frac{\hbar^2}{2m} \frac{3}{3q} + \frac{1}{2} \frac{m \omega^2 q^2}{2} \right] \psi(q, t)
\]

(1.7)

The propagator is function of \(t - t_0\) and for \(t-t_0 < \frac{\pi}{\omega}\), it is given by (*)

\[
K(q, t; q_0, 0) = \left[ \frac{m \omega}{2\pi \hbar \sin \omega t} \right]^{1/2} \exp \left( \frac{im\omega}{2\hbar \sin \omega t} \left[ (q^2 + q_0^2) \cos \omega t - 2qq_0 \right] \right)
\]

(1.8)

The argument of the exponential in (1.8) is \(\frac{1}{\hbar} A(q, q_0, t)\) with \(A\) the classical action for the harmonic oscillator Lagrangian

\[
L(\dot{q}, q) = \frac{1}{2} m \dot{q}^2 - \frac{1}{2} \frac{m \omega^2 q^2}{2}
\]

(1.9)

This is always so for systems with Lagrangian quadratic in \(q\) and \(\dot{q}\).

By developing \(\sin \omega t\) and \(\cos \omega t\) in (1.8) we obtain

(*) For a discussion of the phase factor present for \(t-t_0 > \frac{\pi}{\omega}\), see e.g. \([C 10]\)