MINIMIZATION OF ENERGY FUNCTIONAL WITH CURVE-REPRESENTED EDGES

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1. Introduction

Until the mid eighties almost all edge detection methods were based on local operators. Since the edges of interest usually exist at a coarser scale than the inner scale (corresponding to the pixel size,) of the image, these operators always incorporate some averaging or similar mechanism for suppressing fine scale variations. As a consequence the detected edges get blurred and may be displaced. These problems can be circumvented by applying some kind of nonlinear best fit technique. Such local methods exist in abundance. However, they typically take only a small number of possible local edge configurations into consideration. For all other configurations the blurring and displacement problems thus remain. Ideally one would of course like to consider all possible edge configurations. But then the best fit technique must be applied to image regions so large that the edges therein can be assumed to be sparse. Otherwise the desired fine detail suppression will be difficult, or even impossible, to achieve. Conceptually the simplest way to make sure that this is the case, is to use a global best fit technique, which is the essence of global edge detection or global segmentation.

Quite a few global edge detection methods have been proposed in the computer vision literature to this date. Many of these have more or less been formulated with particular analytical and/or computational techniques in mind. The method proposed in 1985 by Mumford and Shah [263, 264], on the other hand, is based more directly on the desired properties of the solution to the segmentation problem. In this section we outline a rigorous proof of existence of a solution to a slightly restricted version of this problem. Because of space limitations many details will unfortunately have to be left
out. For the complete proof we refer to [274].

1.1. The Mumford and Shah Problem

The global edge detection problem posed by Mumford and Shah can be stated as follows. Given an image, represented by a function \( g \in L_2(B) \), where the image domain \( B \) is some open rectangle in \( \mathbb{R}^2 \), find a closed set \( D \subset \mathbb{R}^2 \) and a function \( f : B \setminus D \to \mathbb{R} \) that minimize the cost or energy functional

\[
\mathcal{C}(D, f) = \int_{B \setminus D} [(f - g)^2 + \mu \|\nabla f\|^2] \, dx + \lambda \text{ length}(D)
\]  

(7.1)

where \( \mu \) and \( \lambda \) are strictly positive constants. The set \( D \) is furthermore assumed to consist of a finite union of \( C^1 \)-smooth curves meeting each other and the image boundary \( \partial B \) only at their endpoints. The original image (function) \( g \) is thereby estimated by a function \( f \), which (if extended to \( B ) \) is smooth in some sense everywhere except on the set \( D \). This set thus represents the edges. It will be referred to as the discontinuity set. Without further restrictions on the original image \( g \), the estimated image (function) \( f \) will naturally be a member of the Sobolev space \( \mathcal{H}^1(B \setminus D) \) consisting of all functions in \( L_2(B \setminus D) \) whose first order distributional derivatives also belong to \( L_2(B \setminus D) \). In the original problem formulation by Mumford and Shah the original image \( g \) was actually assumed to be continuous. With reference to the natural smoothing associated with any physical image acquisition system one could argue that this restriction is realistic. One goal of the imaging system design, however, is to reduce this smoothing to a minimum. Since the continuity assumption also implies that the noise must be continuous, we have simply omitted it in our analysis. This is of course not to say that any conclusions resting on this continuity assumption would be uninteresting.

1.2. The Fixed Discontinuity Set Problem

If the discontinuity set \( D \) is fixed, the last term in the cost (7.1) is constant. The minimization of the total cost \( \mathcal{C}(D, f) \) then reduces to the elliptic problem of minimizing the image cost functional \( \mathcal{C}_\Omega : \mathcal{H}^1(\Omega) \to \mathbb{R} \) defined by

\[
\mathcal{C}_\Omega(f) = \int_{\Omega} [(f - g)^2 + \mu \|\nabla f\|^2] \, dx
\]

Throughout this section the symbol \( \nabla \) will denote the distributional (row vector) gradient operator. The norm \( \| \cdot \| \) (without subscript) is the Euclidean norm (on \( \mathbb{R}^2 \)) and \( \doteq \) indicates equality by definition.