CHAPTER 2

The scalar potential \( \phi \) and the vector potential \( \mathbf{A} \)

2.1. Introduction

Though it is possible to determine the electric and magnetic fields \( \mathbf{E} \) and \( \mathbf{B} \) due to varying charge and current distributions by solving the differential equations (1.125) and (1.122) for \( \mathbf{E} \) and \( \mathbf{B} \), for example using the Jefimenko solutions given by equations (1.136) and (1.134), it is sometimes more convenient to solve problems and to interpret electromagnetism using the scalar potential \( \phi \) and the vector potential \( \mathbf{A} \). Our starting point in this chapter will be Maxwell’s equations for continuous charge and current distributions in otherwise empty space. For these conditions, Maxwell’s equations at a field point inside a charge and current distribution are

\[
\begin{align*}
\nabla \cdot \mathbf{E} &= \frac{\rho}{\varepsilon_0} \quad (2.1) \\
\nabla \cdot \mathbf{B} &= 0 \quad (2.2) \\
\n\nabla \times \mathbf{E} &= -\dot{\mathbf{B}} \quad (2.3) \\
\n\nabla \times \mathbf{B} &= \mu_0 (\mathbf{J} + \varepsilon_0 \dot{\mathbf{E}}) \quad (2.4)
\end{align*}
\]

where \( \rho \) is the charge density and \( \mathbf{J} \) is the current density at the field point. One dot above a variable denotes partial differentiation once with respect to time, two dots above a variable denote partial differentiation twice with respect to time etc. It will be assumed throughout this chapter that there are no dielectrics or magnetic materials so that the relative permittivity \( \varepsilon \), and the relative permeability \( \mu \), are both equal to unity everywhere.

2.2. The differential equations for \( \phi \) and \( \mathbf{A} \)

The vector potential \( \mathbf{A} \) was introduced in Section 1.4.6 of Chapter 1, where we showed, using the Biot-Savart law, that the magnetic field due to a steady current distribution (magnetostatics) could be expressed in terms of a vector
potential $A$ by the equation

$$\mathbf{B} = \nabla \times \mathbf{A}$$

(2.5)

where $\mathbf{A}$ was given by equation (1.74). We shall now go on to consider varying charge and current distributions using Maxwell’s equations as our starting point. It is consistent with equation (2.2) to assume that $\mathbf{B}$ can also be related to a vector potential $\mathbf{A}$ by equation (2.5) in the general case of a varying current distribution. To check this, take the divergence of both sides of equation (2.5) and then use the result that, according to equation (A1.25) of Appendix A1.6, the divergence of the curl of any vector is zero, to show that equation (2.5) leads to equation (2.2). The divergence of $\mathbf{A}$ has yet to be specified.

The scalar potential $\phi$ was first introduced in electrostatics in Section 1.2.9 of Chapter 1, where, in the context of electrostatics, $\phi$ was related to the electrostatic field $\mathbf{E}$ by equation (1.23), according to which

$$\mathbf{E} = -\nabla \phi.$$  

(1.23)

According to equation (A1.26) of Appendix A1.6 the curl of the gradient of any scalar function of position is zero. Hence it follows by taking the curl of both sides of equation (1.23) that equation (1.23) can only be applied in conditions where $\nabla \times \mathbf{E}$ is zero, that is in electrostatics. According to equation (2.3), which expresses Faraday’s law of electromagnetic induction,

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t}.$$  

(2.3)

Substituting for $\mathbf{B}$ using equation (2.5), we have

$$\nabla \times \mathbf{E} = -\frac{\partial}{\partial t} (\nabla \times \mathbf{A}) = -\nabla \times \frac{\partial \mathbf{A}}{\partial t}.$$  

Rearranging,

$$\nabla \times (\mathbf{E} + \dot{\mathbf{A}}) = 0.$$  

(2.6)

Since, according to equation (A1.26) of Appendix A1.6, the curl of the gradient of any scalar function of position is zero it is consistent with equation (2.6) to try putting $(\mathbf{E} + \dot{\mathbf{A}})$ equal to $-\nabla \phi$ in the general case, when the charge and current distributions are varying, giving

$$\mathbf{E} = -\nabla \phi - \frac{\partial \mathbf{A}}{\partial t}.$$  

(2.7)

As a check, integrate equation (2.7) around any closed loop. Since, according to equation (A1.11) of Appendix A1.2, $\int \nabla \phi \cdot d\mathbf{l}$ is always zero, we have

$$\oint \mathbf{E} \cdot d\mathbf{l} = -\oint \nabla \phi \cdot d\mathbf{l} - \oint \frac{\partial \mathbf{A}}{\partial t} \cdot d\mathbf{l} = -\frac{\partial}{\partial t} \oint \mathbf{A} \cdot d\mathbf{l}.$$  

Applying Stokes’ theorem, equation (A1.34) of Appendix A1.8, to $\oint \mathbf{A} \cdot d\mathbf{l}$ and putting $\nabla \times \mathbf{A}$ equal to $\mathbf{B}$ we obtain...