Chapter 7

DIRECT NUMERICAL METHODS FOR OPTIMAL CONTROL PROBLEMS

Development of interior point methods for linear and quadratic programming problems occurred during the 1990’s. Because of their simplicity and their convergence properties, interior point methods are attractive solvers for such problems. Moreover, extensions have been made to more general convex programming problems.

The link from optimal control problems to programming problems is straightforward: a discretization method (for instance, a Finite Element Method) is applied, leading to a mathematical programming problem which can be handled by an interior point algorithm. We therefore obtain a class of direct methods for optimal control problems.

We shall consider quadratic optimal control problems with bound constraints for the control and for the state. The Finite Element Method leads to large scale quadratic programming problems. Interior point methods are iterative; in each iteration they generate approximations to the solution that are strictly feasible with respect to the bound constraints. The main subproblem in each iteration is to solve a linear system which is large, indefinite and ill-conditioned.

An abstract quadratic optimal control problem, together with examples, is introduced in Section 7.1. The FEM is presented in Section 7.2, and the interior-point methods for the quadratic programming problem in Section 7.3. Section 7.4 deals with the solution of the linear system by Krylov subspace algorithms. Their convergence properties and their implementation are discussed. The subject of Section 7.5 is the preconditioning for the algorithms in the previous section and also for the Karush–Kuhn–Tucker matrices which arise from optimality conditions.

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1. The abstract optimal control problem

The abstract functional framework is given by the real Hilbert spaces $V$, $H$, and $U$. We denote by $\langle \cdot, \cdot \rangle_H$ the inner product of $H$ and by $\langle \cdot, \cdot \rangle$ the inner product of $U$. The norms are distinguished by using subscripts. We suppose that $V \subset H$, the inclusion being dense and compact, and that

$$\|v\|_H \leq \|v\|_V \quad \text{for any } v \in V. \quad (7.1)$$

$H$ and $U$ are identified with their respective duals. $K_u$ is a nonempty closed convex set in $U$, and $K_y$ is a nonempty closed convex set in $V$.

Let $a : V \times V \to \mathbb{R}$ be a bilinear, symmetric, continuous, and $V$-elliptic form; there exists a constant $\alpha > 0$ such that

$$a(v, v) \geq \alpha \|v\|^2_V \quad \text{for any } v \in V. \quad (7.2)$$

Consider also $b : U \times H \to \mathbb{R}$ a bilinear and continuous form; there is a constant $\beta > 0$ such that

$$|b(u, v)| \leq \beta \|u\|_U \cdot \|v\|_H \quad \text{for any } u \in U \text{ and } v \in H. \quad (7.3)$$

Moreover, $\ell : H \to \mathbb{R}$ is a linear and continuous functional, $y_d \in V$ is a given element, and $\gamma > 0$ is a given constant.

Introduce the optimal control problem

(P) Minimize

$$\Phi(u, y) = \frac{1}{2} \|y - y_d\|^2_H + \frac{\gamma}{2} \|u\|^2_U$$

for $u \in K_u$ and $y \in K_y$ subject to the state equation

(SE) \quad $a(y, v) + b(u, v) = \ell(v) \quad \text{for any } v \in V.$

It is well known that problem (P) has a unique optimal pair. For results on optimal control problems governed by elliptic variational inequalities, refer to Barbu, 1993, Chapter 3.

Example 7.1 Let $\Omega \subset \mathbb{R}^N$, $N = 2, 3$, be a convex polyhedral open set, and consider the (state) equation

$$\begin{cases}
-\Delta y = u + f & \text{in } \Omega, \\
y = 0 & \text{on } \partial \Omega.
\end{cases}$$

In the 2D case this corresponds to the problem of a membrane fixed at its boundary and loaded by a transverse force $(u(x) + f(x))dx$ on the surface element $dx$ (e.g., Ciarlet, 1994, Section 3.2).