STATISTICAL ESTIMATION OF PARAMETERS OF THE WEIBULL AND FRECHET DISTRIBUTIONS

Nancy R. Mann
Department of Biomathematics, UCLA
Los Angeles, CA 90024

ABSTRACT: Estimation procedures for parameters of the Frechet and the three-parameter Weibull distribution are reviewed, after the relationships between these two distributions and the Gumbel distribution are discussed. Included in the review are maximum-likelihood and moment estimators as well as linearly based estimators and some that are extremely simple, involving only a few order statistics. Large- and small-sample properties are discussed.

INTRODUCTION

The Weibull distribution is an asymptotic distribution of smallest extremes, with the initial underlying distribution being bounded below. Thus, any Weibull variate, X, is larger than some lower threshold value λ, which may be equal to zero. Early investigators such as Fisher and Tippett (1928), who originally derived the distribution called the Weibull a "type-III asymptotic distribution" of smallest extremes. Gnedenko (1943) and certain others designated it "type-II".

Waloddi Weibull (1939) gave an empirical derivation of this distribution in an analysis of dynamic breaking strengths by requiring only that it meet certain practical criteria. That article and other related articles of Weibull dealing with the modeling of failure (or survival) data seem to have found a large audience among those concerned with reliability analysis, particularly after the advent of the space age. Thus, the name Weibull has become firmly attached to the distribution, and its use as a model for "time-to-failure" has been widespread. It arises naturally also

81

in the analysis of droughts. See, for example, Gumbel (1954).

The mirror image of the Weibull distribution, the Fisher-Tippett type-III distribution of largest extremes, has been used to model maximum temperatures, maximum wind speeds and maximum earthquake magnitudes. See Jenkinson (1955) and Yegulalp and Kuo (1974).

The distribution function of a Weibull variate $X$ with threshold (or location) parameter $\lambda$ is given by

$$
F_X(x) = \begin{cases} 
1 - \exp\left[-\left(\frac{x-\lambda}{\delta}\right)^\beta\right], & x \geq \lambda \\
0, & \text{otherwise.}
\end{cases}
$$

The parameters $\beta > 0$ and $\delta > 0$ are associated with shape and scale, respectively. If $\lambda$ is equal to zero, then $Z = \ln X$ has a Gumbel, or Fisher-Tippett Type-I, distribution with location parameter $\eta = \ln \delta$ and scale parameter $\xi = 1/\beta$. Many estimation procedures for the two-parameter Weibull have been developed as techniques for estimating location and scale parameters of the Gumbel distribution. See Mann, Schafer, Singpurwalla (1974) and Mann and Singpurwalla (1982).

The Frechet distribution is an asymptotic distribution of largest extremes, derived by Frechet under the condition that the initial variates be non-negative. The distribution function for a Frechet variate $Y$ has the form

$$
F_Y(y) = \begin{cases} 
\exp\left[-\left(\frac{Y}{\gamma}\right)^\alpha\right], & y > 0 \\
0, & \text{otherwise,}
\end{cases}
$$

with $\alpha > 0$ a shape parameter and $\gamma > 0$ a scale parameter.

The Frechet distribution is a special case of a Fisher-Tippett type-II distribution of largest extremes, i.e., one with threshold parameter $\mu$ equal to zero. In the case of $\mu \neq 0$, $y$ is replaced by $y - \mu$ in the right side of Equation (2). The Frechet distribution has been widely used as a model for floods and maximum rainfalls. See Gumbel (1954) and Jenkinson (1955).

It is fairly evident that if $Y$ has a Frechet distribution with shape and scale parameters $\alpha$ and $\gamma$, respectively, then $Y^{-1}$ has a two-parameter Weibull distribution with shape parameter $\alpha$ and scale parameter $\gamma^{-1}$. Thus, when data have not been censored, estimation procedures that have been derived for the two-parameter Weibull distribution can be used to estimate parameters of the Frechet distribution by simply taking reciprocals of the observed values. Methods applicable for Weibull censoring on the right