Chapter 7

SPLINE AND CIESIELSKI SYSTEMS

7.1. Introduction

In the first four sections we consider the Kronecker product of the unbounded Ciesielski or spline systems \((h_{mj,kj})\) of order \((mj,kj)\) introduced by Ciesielski [38, 45, 46] \(|kj| \leq mj + 1, mj \geq -1\) and the Hardy spaces \(H_{p,q} = H_{p,q}^{(mj,kj)}\) and \(H_{p,q}^\square = H_{p,q}^{\square,(mj,kj)}\) defined by the \(L_{p,q}\) norms of the non-tangential maximal functions. Note that if \(kj = mj + 1\) for all \(j\) then we obtain the usual dyadic Hardy spaces, in other cases \(H_{p,q}\) is the restriction of the classical \(H_{p,q}\) space to the unit cube. It is known that \((h_{mj,kj})\) and \((h_{mj,-kj})\) are biorthogonal systems and that \((h_{mj,kj})\) is the Haar system if \(mj = -1, kj = 0\), the Franklin system if \(mj = 0, kj = 0\).

In Section 7.2 we give the basic definitions and introduce the following operators. The conditional partial sums are defined by

\[ T_n^{(m,k)}f = \sum_{j \leq n} \epsilon_j (f, h_j^{(m,k)}) h_j^{(m,-k)} \]

where \(\epsilon_j = \pm 1\) \((j \in \mathbb{N}^d)\) and \((\cdot, \cdot)\) denotes the usual scalar product. If all \(\epsilon_j = 1\) then we get the partial sums \(P_n^{(m,k)}\) and we define the maximal operators \(P_n^{(m,k)}\) and \(P_n^\square^{(m,k)}\) in the usual way.

In the one-dimensional case Ciesielski [45, 46, 42] proved that the operators \(T_n^{(m,k)}\) and \(P_n^{(m,k)}\) are bounded on the \(L_p[0,1]\) spaces \((1 < p < \infty)\) and are of weak type \((L_1, L_1)\), uniformly in \(n\). Schipp and Simon [155] verified that these operators are bounded from the Hardy space \(H_1[0,1]\) to \(L_1[0,1]\) whenever \(k = 0\) and \(m = -1, 0\), i.e. for the Haar and Franklin systems.
In Sections 7.3 and 7.4 we generalize these results for the product setting. First we show that the operators $P_{\Box}^{(m,k)}$ and $P_{\square}^{(m,k)}$ are bounded on $L_p[0,1]^d$ $(1 < p < \infty)$. Then we prove that $P_{\Box}^{(m,k)}$ is bounded from the Hardy space $H_p^\Box$ to $L_p[0,1]^d$ $(d/(d + 1) < p < \infty)$ and is of weak type $(L_1, L_1)$. Moreover, $P_{\square}^{(m,k)}$ is bounded from the Hardy space $H_p$ to $L_p[0,1]^d$ $(1/(m_j - k_j + 2) < p < \infty)$ and is of weak type $(H_1^p, L_1)$. As a consequence we obtain that $P_n^{(m,k)} \rightarrow f$ a.e. if $f \in H_1^p$ as $n \rightarrow \infty$ and if $f \in L_1$ as $n \rightarrow \infty$ in a cone.

Bočkariev [14] proved that the Franklin system is an unconditional basis in $L_p[0,1)$ $(1 < p < \infty)$. Some years later Ciesielski [46] verified that the spline systems are equivalent and unconditional bases in $L_p[0,1)$ $(1 < p < \infty)$. Maurey [128] proved that the classical $H_1$ space has an unconditional basis. His proof is non-constructive, he proved that $H_1$ is linearly isomorphic to the dyadic $H_1$ in which the Haar system is an unconditional basis. Carleson [29] constructed an unconditional basis in $H_1$ and Wojtaszczyk [259] verified that the Franklin system is an unconditional basis in $H_1$. Later Sjölin and Strömberg [183] and Wojtaszczyk [260] extended these results to Hardy spaces, they obtained that the spline systems are equivalent and unconditional bases also in $H_p[0,1)$ $(1/(m - k + 2) < p \leq 1)$.

In Section 7.5 we will extend these results also to the product setting. The unconditional basis property of the Kronecker product of the spline systems in $L_p[0,1]^d$ follows by iteration from the one-dimensional result. We show that $T_n^{(m,k)}$ is bounded from the Hardy space $H_p$ to $L_p[0,1]^d$ $(1/(m_j - k_j + 2) < p < \infty)$. From this it follows that the $p$-norm of the square function can also be estimated by the Hardy norm. Using a result of Sjölin and Strömberg [183] we prove the equivalence between the $p$-norm of the square function and the Hardy norm, which implies that the multi-parameter spline functions are equivalent and unconditional bases in $H_p$ $(1/(m_j - k_j + 2) < p < \infty)$. We give also a nice characterization of the Hardy spaces, a function belongs to $H_p$ if and only if $P_n^{(m,k)}f \rightarrow f$ unconditionally in $L_p$ norm as $n \rightarrow \infty$, which is due to Chang and Ciesielski [30] in the one-dimensional case and $p = 1$.

The bounded Ciesielski systems can be obtained from the spline systems of order $(m, k)$ in the same way as the Walsh system from the Haar one (see Ciesielski [43, 42, 46]). So the bounded Ciesielski systems can be regarded as generalizations of the Walsh system. Ciesielski [46, 42] proved that the maximal operator of the Fourier series with respect to these bounded Ciesielski systems is bounded on $L_p[0,1)$ $(1 < p < \infty)$ and so the Fourier series of a function $f \in L_p[0,1)$ converges to $f$ a.e. and