CHAPTER 22

THE CODING OF TREES

REFLECTION PRINCIPLES

A reflection principle is a sequent of the form \([(\zeta) \, DT(U, [ \rightarrow C I]) \rightarrow C I^X],\)
where \(X\) are irreducible terms. (Anything enclosed within ' [...]' brackets is a sequent.)

CODING OF PROTOLOGICAL DERIVATION TREES

A derivation tree (from a given list of premises \(X_0, \ldots X_{k-1}\)) is a tree of sequents whose leaves are protological axioms, reflection principles, and premises from the given premise list, and whose other nodes are related by the protological inference rules. A derivation (from a given premise list) is a derivation tree with its conclusion sequent (the root of the tree) removed.

Derivations and derivation trees may be coded as irreducible terms, using five 1-ary constructors premise, none, one, two and rp. The free variables of the code of a derivation or derivation tree are the free variables of its constituent sequents.

THE DERIVATION TREE PREDICATE, \(DT\)

Four constructions are defined: \(DT, \mathcal{E}, join\) and \(Join\).

THEOREM 1. \(DT(\mathcal{E}, [ \rightarrow T]) \triangleright^* T\).

THEOREM 2. \(DT(rp(\lambda[z].U), (\zeta) \, DT(U, [ \rightarrow C I]) \rightarrow C I) \triangleright^* true\).

TREE CODES AND WELL-FOUNDEDNESS

DEFINITION. A type symbol is a construction that has been given an interpretation as representing a coding of a class of trees as constructions.

A well-foundedness relation \(T : \Pi\) is defined, where \(T\) is any term and \(\Pi\) is a type symbol (or a term reducing to a type symbol).
From now on the identifiers ‘\(\Pi\)’, ‘\(\Sigma\)’, ‘\(\Phi\)’, ‘\(\Psi\)’, ‘\(\Omega\)’ and ‘\(\Theta\)’, sometimes with subscripts or primes, will always be used as metavariables denoting type symbols (or terms that reduce to type symbols).

**THEOREM 3.** (Well-foundedness reduction rule.)

- If \(A \triangleright^* \triangleleft B\) then \(A : \Pi\) iff \(B : \Pi\).
- If \(\Pi \triangleright^* \triangleleft \Sigma\) then \(A : \Pi\) iff \(A : \Sigma\).

Two fresh 0-ary constructors, \(\text{leaf}\) and \(\text{rt}\), are introduced, along with three fresh 1-ary constructors, \(\text{map}\), \(\text{product}\) and \(\text{pi}\). The following type symbols are defined: \(\text{leaf} \), \(\text{map}(\Pi, \Sigma)\), \(\text{product}(\Pi, \Sigma)\), \(\text{pi}[\Phi_1, \ldots, \Phi_k]\), and \(\text{rt}\), where \(\Pi, \Sigma, \Phi_1, \ldots, \Phi_k\) are type symbols.

**THEOREM 4.** (\(\text{leaf}\) rule.) For any term \(T\), \(T : \text{leaf}\).

**THEOREM 5.** (\(\text{map}\) rule.)

- If \(A : \text{map}(\Pi, \Sigma)\) and \(B : \Pi\) then \(AB : \Sigma\).
- If, for any construction \(B : \Pi\), \(AB : \Sigma\), then \(A : \text{map}(\Pi, \Sigma)\).
- If \(X \not\in u\), \(u \not\in X\), \(U\) are constructions, and \(T[U/u] : \Sigma\), then \((\lambda X.T)[U/u] : \text{map}(\text{leaf}, \Sigma)\).

**THEOREM 6.** (Well-foundedness instantiation rule.) If \(A : \Pi\) and \(X \not\in\) then \(A[X/x] : \Pi\).

**THEOREM 7.** (\(\text{product}\) rule.)

- Any construction \(T\) such that \(T : \text{product}(\Pi, \Sigma)\) is of the form \((X, Y)\) for some constructions \(X : \Pi\) and \(Y : \Sigma\).
- \(\text{left} : \text{map}(\text{product}(\Pi, \Sigma), \Pi)\).
- \(\text{right} : \text{map}(\text{product}(\Pi, \Sigma), \Sigma)\).
- For any terms \(A\) and \(B\), if \(A : \Pi\) and \(B : \Sigma\) then \((A, B) : \text{product}(\Pi, \Sigma)\).

**THEOREM 8.** (\(\pi\) rule.)

- Any construction \(T\) such that \(T : \pi[\Pi, \ldots, \Sigma]\) is of the form \([A, \ldots, B]\) for some constructions \(A : \Pi, \ldots, B : \Sigma\).
- \(\text{left} : \text{map}(\text{product}(\Pi, \Sigma), \Pi)\).
- For any terms \(A, \ldots, B\), if \(A : \Pi, \ldots, B : \Sigma\) then \([A, \ldots, B] : \pi[\Pi, \ldots, \Sigma]\).

**THEOREM 9.** (\(\text{if}\) well-foundedness rule.) \(\text{if} : \text{map}(\text{leaf}, \text{map}(\Pi, \text{map}(\Pi, \Pi)))\).

**THEOREM 10.** (Recursion well-foundedness rule.)

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\text{rec}_0 : \text{map}(\text{product}(\Pi, \text{map}(\text{product}(\text{leaf}, \Pi), \Pi)), \text{map}(\text{leaf}, \Pi))\]

**THEOREM 11.** (\(\text{fxpt id}\) well-foundedness rule.) \(\text{fxpt id} : \text{map}(\Pi, \Sigma)\).