8 THE RECURSION FORMULAE FOR THE GAMMA DISTRIBUTION

8.1 INTRODUCTION

The gamma distribution is—as the normal and the log normal distributions—often used to describe the demand for a product property. We mention for instance the demand for the thickness of window panes which follows approximately a gamma distribution [AKB, (1976)].

The gamma distribution has two parameters. By changing the scale of measurement on the x-axis, one of these parameters can be eliminated. We have chosen the transformation such that the transformed gamma variable has a density function which equals that of a $\chi^2$ distribution. Note however, that this does not imply that the transformed gamma variable has a $\chi^2$ distribution; the $\chi^2$ distribution namely is only defined for integer values of its parameter, usually denoted by the term “degrees of freedom,” while we will allow it to have fractional values too. The main reason for the choice of this particular transformation is that integrals of the $\chi^2$ distribution are extensively tabulated for integer values of its parameter. This implies that eventually optimal size patterns might be calculated—or approximated if the parameter is fractional—with a simple calculator instead of with a computer.

Of special interest, from the mathematical point of view, is the exponential
distribution which is a special case of the gamma distribution. This distribution permits, as we will show, an analytical solution of the optimal size pattern if the linear loss function is used. Besides this, we will prove for the specific case that the linear loss function is used in combination with shift rule O, that the expected adaptation loss reaches a minimum at the size pattern satisfying the recursion formulae.

8.2 THE PROBABILITY DENSITY FUNCTION

The probability density function of a gamma distributed variable is,

\[ g(x) = \frac{1}{\Gamma(a)b^a} x^{a-1} e^{-x/b} \quad x > 0, \ a, b > 0, \]

\[ = 0 \quad \text{elsewhere}, \] (8.1)

where

\[ \Gamma(a) = \int_0^{\infty} t^{a-1} e^{-t} dt. \] (8.2)

By virtue of (8.2),

\[ \Gamma(a) = (a - 1) \Gamma(a - 1) \] (8.3)

and, since \( \Gamma(1) = 1 \), we may write for integer values of \( a \)

\[ \Gamma(a) = (a - 1)! \] (8.4)

Furthermore, according to (4.1) and (4.2), \( x_0 \) equals zero and \( x_{n+1} \) equals infinity. The mean and variance of a gamma distributed variable are

\[ E(x) = ab, \] (8.5)

\[ V(x) = ab^2. \] (8.6)

If \( x \) has a gamma distribution with parameters \( a = r/2 \) and \( b \), where \( r \) is a positive integer, then the random variable \( v \),

\[ v = \frac{2x}{b} \] (8.7)

has a \( \chi^2 \) distribution with \( r \) degrees of freedom. The probability density function of \( v \) is;

\[ h(v) = \frac{1}{\Gamma(r/2)2^{r/2}} v^{r/2-1} e^{-v/2} \quad v > 0, \ r > 0, \]

\[ = 0 \quad \text{elsewhere}. \] (8.8)