A central crack of mode III in a rectangular sheet with fixed edges

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Received 22 June 1987; accepted 13 January 1988

Abstract. This paper discusses the general solution, by use of the Fourier transform and Fourier series, of the stress intensity factor in a rectangular sheet containing a central crack of mode III, where its boundary is constrained, but its crack surface is subjected to an arbitrary anti-plane load. As examples of numerical computing, consider three circumstances: (1) the boundary parallel to a central crack is fixed; (2) the boundary perpendicular to a central crack is fixed; (3) all of the boundaries around a central crack are fixed but the crack surface is subjected to a constant load.

1. Introduction

Both [1] and [2] have given the solutions of a finite rectangular sheet containing a central crack of mode III where its boundary is free or is subjected to an arbitrary anti-plane load. In the present study, we will consider the general solution of the stress intensity factor in a rectangular sheet with a central crack of mode III, where its boundary is constrained, but its crack surface is subjected to an arbitrary anti-plane load.

The solution of the problem is obtained by means of the Fourier transform and Fourier series. In Section 2, we have described this method in detail and considered the problem of a rectangular sheet where its boundary parallel to its central crack, is fixed as shown in Fig. 1(a). In Section 3, the other circumstances with fixed edges have been discussed, including a rectangular sheet with fixed edges perpendicular to its central crack as in Fig. 1(b), and with four fixed edges as shown in Fig. 1(c).

The numerical solutions of the stress intensity factor under the three circumstances are given in Tables 1 and 2 as the anti-plane shear stress is a constant \( \tau \). However, if \( \tau \) is changeable along with a central crack surface, i.e., \( \tau = \tau(x) \), the solutions can be obtained in the same way.

2. A rectangular sheet with fixed edges parallel to its central crack

Consider the finite rectangular sheet \((2h \times 2b)\) with a central crack of length \(2a\) and its upper and lower surface \(y = \pm h\) to be free of displacement as shown in Fig. 1(a). Assume the crack is subjected to an arbitrary anti-plane load \(\tau(x)\) forming a self-equilibrating system and for convenience suppose the load to be symmetrical to the \(y\)-axis. The only nonzero displacement component \(U_z\) in the \(z\) direction for this problem may be specified as follows:

\[
\frac{\partial^2 U_z(x, y)}{\partial x^2} + \frac{\partial^2 U_z(x, y)}{\partial y^2} = 0
\]  

(2.1)
according to the symmetry with boundary conditions

\[
U_z(x, h) = 0, \quad 0 \leq x \leq b, \tag{2.2}
\]

\[
\tau_{xz}(b, y) = \mu \frac{\partial U_z(b, y)}{\partial x} = 0, \quad 0 \leq y \leq h, \tag{2.3}
\]

\[
\tau_{yz}(x, 0) = \mu \frac{\partial U_z(x, 0)}{\partial y} = -\tau(x), \quad 0 \leq x < a, \tag{2.4}
\]

\[
U_z(x, 0) = 0, \quad a < x \leq b, \tag{2.5}
\]

where \(\mu\) stands for the shear modulus.

By means of the Fourier cosine transform and the Fourier sine series, the solution \(U_z(x, y)\) of (2.1) is

\[
U_z(x, y) = \frac{1}{\mu} \left\{ \int_0^\infty A(w) \frac{\sin(w(h-y))}{\sin(wb)} \cos(wx) \, dw + \sum_{n=1}^{\infty} B(n) \frac{\sin(n\pi y)}{n\pi h} \right\}. \tag{2.6}
\]

Comparing (2.2) and (2.6), the boundary condition (2.2) is evidently satisfied. With the aid of the boundary condition (2.3), the relation between \(A(w)\) and \(B(n)\) is as follows:

\[
\sum_{n=1}^{\infty} B(n) \frac{n\pi h}{b} \frac{\sin(n\pi y)}{n\pi h} = \int_0^\infty w A(w) \frac{\sin(w(h-y))}{\sin(wb)} \, dw. \tag{2.7}
\]

Employing the Fourier sine series pair,

\[
f(y) = \sum_{n=1}^{\infty} B(n) \sin\frac{n\pi y}{h}; \tag{2.8}
\]

\[
B(n) = \frac{2}{h} \int_0^h f(y) \frac{n\pi y}{h} \, dy.
\]