Cruciform crack in an orthotropic elastic plane

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Abstract. The problems of determining the stress and displacement fields in an infinite orthotropic plane containing a cruciform crack \(|x| \leq a, y = 0\) and \(|y| \leq b, x = 0\), when (I) the shape of the crack is prescribed and (II) the cracks are opened by given normal pressures, are reduced to mixed boundary value problems for the quarter plane. Using integral transform techniques, a closed form solution is obtained for problem I, whereas the solution of problem II has been reduced to solving a Fredholm integral equation of second kind with non-singular kernel. Numerical calculation of the stress intensity factor and crack energy in the case of a linear loading function for various crack lengths are presented for problem II, using the values of material constants for a Boron-Epoxy composite.

1. Introduction

The static problem of determining the distribution of stress and displacement in an isotropic elastic plane containing a cruciform crack, a special case of the star-shaped crack, has been considered by Sneddon [1], Stallybrass [2], Rooke and Sneddon [3], Sneddon and Das [4] and many others, by use of Muskhelishvilli-Kolosov potential functions, integral transform techniques and the Wiener-Hopf technique. Recently, Ong and Srivastav [5] and Brock and Deng [6] considered dynamic problems associated with the cruciform crack in an isotropic elastic medium. Such crack problems for anisotropic media in general and orthotropic media in particular are important. Dhaliwal [7], Satapathy and Parhi [8], Kushwaha [9], Das and Behera [10] and Konishi and Atsumi [11] have considered Griffith cracks in orthotropic media. In this paper we have considered two static cruciform crack problems in an infinite orthotropic elastic plane; one of them, (I) having a prescribed crack shape and the other, (II) having been opened by prescribed normal pressures. Integral transform technique is employed to solve the problems. A closed form solution is obtained for problem I. The solution of problem II has been reduced to solving a Fredholm integral equation of second kind which is suitable for numerical computation.

In problem I, expressions for the pressure necessary to produce the crack shapes have been obtained. In problem II, expressions for quantities of physical interest, e.g. shape of the crack, stress intensity factor and crack energy have been obtained.

The numerical computation in problem II has been done for Boron-Epoxy composite material and linearly loaded cracks.

2. The basic equations

For an orthotropic medium, we choose the Cartesian coordinate axes to coincide with the principal material axes and define the plane strain state (c.f. Satapathy and Parhi [8]) by

\[
\begin{align*}
  u_x &= u(x, y), & u_y &= v(x, y), & u_z &= 0,
\end{align*}
\]
where $u_x, u_y, u_z$ are displacements in $x, y$ and $z$ directions. The stress-displacement relations are

$$
\sigma_{xx} = A_{11} \frac{\partial u_x}{\partial x} + A_{12} \frac{\partial u_y}{\partial y},
$$

$$
\sigma_{xy} = A_{12} \frac{\partial u_x}{\partial x} + A_{22} \frac{\partial u_y}{\partial y},
$$

$$
\sigma_{yz} = A_{66} \left( \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial y} \right),
$$

$$
\sigma_{xz} = 0 = \sigma_{zx},
$$

where $A_{ij}$ are anisotropic constants of the orthotropic material. The equations of equilibrium in terms of displacements are

$$
A_{11} \frac{\partial^2 u_x}{\partial x^2} + A_{66} \frac{\partial^2 u_x}{\partial y^2} + (A_{12} + A_{66}) \frac{\partial^2 u_y}{\partial x \partial y} = 0,
$$

$$
A_{66} \frac{\partial^2 u_y}{\partial x^2} + A_{22} \frac{\partial^2 u_y}{\partial y^2} + (A_{12} + A_{66}) \frac{\partial^2 u_x}{\partial x \partial y} = 0.
$$

(2.3)

We assume that the solution may be found in the form

$$
u_x = \frac{\partial \phi}{\partial x}, \quad u_y = \lambda \frac{\partial \phi}{\partial y}
$$

(2.4)

where $\lambda$ is a constant and $\phi$ is a function of $x$ and $y$. The equations in (2.3) are satisfied if

$$
\frac{A_{66} + \lambda(A_{12} + A_{66})}{A_{11}} = \frac{\lambda A_{22}}{\lambda A_{66} + A_{12} + A_{66}} = \delta^2,
$$

(2.5)

where $\delta^2$ is another constant. This yields two quadratic equations, one in $\lambda$ and the other in $\delta^2$. $\lambda_1, \lambda_2$ and $\delta_1^2, \delta_2^2$ are the roots of the quadratic equations

$$
\lambda^2 A_{66}(A_{12} + A_{66}) + \lambda[(A_{12} + A_{66})^2 + A_{66}^2 - A_{11}A_{22}] + A_{66}(A_{12} + A_{66}) = 0
$$

(2.6)

and

$$
\delta^4 A_{11} A_{66} + \delta^2[(A_{12} + A_{66})^2 - A_{66}^2 - A_{11}A_{22}] + A_{22} A_{66} = 0
$$

(2.7)

respectively. Hence, we have two potential functions $\phi_1$ and $\phi_2$ satisfying the differential equation

$$
\frac{\partial^2 \phi_i}{\partial x^2} + \frac{\partial^2 \phi_i}{\partial y^2} = 0, \quad (i = 1, 2),
$$

(2.8)