An elastic-plastic finite element analysis of crack tip fields under biaxial loading conditions

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ABSTRACT
A square plate containing a central crack and subjected to biaxial stresses has been studied by a finite element analysis. An elastic analysis shows that the crack opening displacement and stress of separation ahead of the crack tip are not affected by the mode of biaxial loading and therefore the stress intensity factor adequately describes the crack tip states in an elastic continuum.

An elastic-plastic analysis involving more than localized yielding at the crack tip provides different solutions of crack tip stress fields and crack face displacements for the different modes of biaxial loading.

The equi-biaxial loading mode causes the greatest separation stress but the smallest plastic shear ear and crack displacement. The shear loading system induces the maximum size of shear ear and crack displacement but the smallest value of crack tip separation stress.

Notation

\( \sigma \) tensile stress

\( \nu \) Poisson’s ratio

\( E \) Young’s modulus of elasticity

\( m \) plastic modulus

\( l \) distance ahead of the crack tip

\( d \) distance from the crack tip to a point in crack space

\( \tau_p \) yield shear stress for plane strain \((3^{-\frac{1}{2}} \text{ yield stress in tension})\)

\( K_1 \) Irwin stress intensity factor; mode I.

\( K^* \) implied, non-dimensional, stress intensity factor

\( v \) the displacement of a crack face due to the application of stress

\( a \) half crack length

\( \Delta \) the ratio \( v/\sigma_P \) where \( \sigma_P \) is the stress applied normal to the crack plane

\( \delta \) ratio of the maximum value of displacement, \( v \), and the half crack length, \( a \)

\( r \) the ratio defining the biaxial stress mode, \( \sigma_Q/\sigma_P \).

1. Introduction

In recent years one main stream of fracture studies has involved the determination of a crack tip stress intensity factor, \( K \), a parameter which helps define the stress field ahead of the crack tip [1]. This approach has found universal appeal for cracks in bodies that deform elastically and is found to be particularly useful in the design of structures containing geometric discontinuities.

For post-yield conditions the crack opening displacement (COD) approach has been used and is especially valuable in the experimental field since the displacement is a function of the elastic behaviour, yield strain and work hardening characteristic of the material being studied [2]. In recent times the COD technique has also been applied to pre-yield conditions [3].

Most work however is concentrated on the study of cracks in a uniaxial stress system where the plane of the crack is perpendicular to the direction of the remotely applied tensile force,
It is considered that a tensile stress, \( \sigma_p \), remotely applied in a direction perpendicular to the line of the crack tip and \( \sigma_p \) will have no effect in the crack tip zone on the stresses normal to the crack plane. For an elastic continuum this would be the case but for real materials a biaxial stress system will induce a different state at the crack tip compared to a uniaxially applied force. Unfortunately no work has been published on the effect of biaxial stress systems on \( K_1 \) and COD values for an elastic-plastic material.

Now in engineering components, cracks, notches etc. are invariably located in a multiaxial strain field and such geometrical discontinuities can and do lead to fatigue failures. A recent theory has been proposed for multiaxial fatigue failure based on strain parameters [4], [5] and the present work is an attempt to understand fatigue crack growth in a biaxial stress-strain field of an elastic-plastic material using a finite elements program.

2. The program

The finite elements program was written for the University of Cambridge Atlas II computer (TITAN). It is based on the incremental load, initial stress, compatible displacement approach [6] and uses for its idealization the simplest of the family of isoparametric quadrilateral elements which are capable of admitting some linear variation of the strain within the elements. On the application of load increments, the stresses at the corners are sensed for yielding and any plastic strain increments which occur are evaluated using a yield criterion and a plastic flow rule. The self-equilibrating initial equivalent nodal forces are calculated by interpolating the plastic strain in the region of the element, using the shape functions for the element. During each loading step, iterations are carried out, accelerated by an over-relaxation factor, until both equilibrium and the yield criterion are satisfied. The von Mises yield criterion for three dimensions is adopted for this program which when translated to two-dimensional plane strain conditions involves a variable effective Poisson’s ratio \( v^* \) that may take values in the range \( v \leq v^* \leq \frac{1}{2} \), where \( v \) is the elastic Poisson’s ratio for the material. The analysis accommodates changes of geometry caused by finite displacements. The program can accept zero or finite values of strain hardening coefficients and other program options include plane stress or axisymmetry conditions as well as provision for thermal stresses in elastic-plastic situations*.

3. The analysis

The general biaxial stress system which is analyzed in this paper is shown in Fig. 1. The three

![Figure 1](image-url)

Figure 1. Centre cracked plate subjected to biaxial stresses.

* Independent analyses of the same notched bar, subjected to pure bending, by this program [6] slip line theory [7] and other programs [8] are in good agreement and provide a useful comparison of the different methods. The results will appear in a future publication [9].

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