The partially closed Griffith crack

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ABSTRACT
A solution is presented for a Griffith crack subjected to an arbitrary polynomial loading function which causes one end of the crack to remain closed. Closed form expressions are presented for the crack opening length and for the stress and displacements in the plane of the crack. The special case of pure bending is presented as an example and for this case the stress intensity factor is computed.

Nomenclature

\( \sigma_{ij}, \tau_{ij} \)  stress components
\( U \)  x displacement
\( V \)  y displacement
\( a \)  length of open portion of the crack
\( a_0 \)  physical crack length
\( P(y) \)  load on the crack surface
\( \nu \)  Poisson ratio
\( \beta^2 \)  \( 2(1-\nu)/(1-2\nu) \)
\( G \)  shear modulus
\( \phi \)  potential function
\( P_s(y) \)  symmetric portion of \( P(y) \)
\( P_a(y) \)  anti-symmetric portion of \( P(y) \)
\( \alpha_n \)  loading coefficients for the polynomial \( P(y') \)
\( \beta_m \)  loading coefficients for the polynomial \( P(y) \)
\( J_\nu(z) \)  Bessel function of the first kind of order \( \nu \)
\( K_I \)  stress intensity factor
\( x, y \)  two-dimensional rectangular Cartesian coordinates

1. Introduction

There are many solutions available for two-dimensional crack problems in linear elastic fracture mechanics which involve the requirement that the loading causes the crack surfaces to move apart. This paper presents a solution for a two-dimensional through crack in which the requirement of crack surface separation has been relaxed. This type of fracture problem does not appear to have been discussed to any great extent in the literature. Burniston [1] solved the partially closed Griffith crack for the case where the crack is closed at its center by the application of concentrated forces located at points within the solid above and below the centerline of the crack. Tweed [5] has derived formulae for the crack opening shape and stress intensity factor for a partially closed Griffith crack in an infinite elastic solid which is loaded in biaxial tension and symmetric body forces. The solution presented here is applicable to the more practical problem of a partial closure of one end of a Griffith crack due to the presence of a
nonuniform applied stress field in the region of the crack. As an example of this situation, results are presented for a crack centrally located in a large beam in pure bending.

2. The problem

Figure 1 illustrates the problem under consideration showing the crack in a partially closed configuration resulting from the application of an arbitrary nonuniform stress. In Figure 1 "\(a_0\)" represents the half length of the actual crack while "\(a\)" represents the half length of the open portion of the crack. The loading \(P(y)\) is the stress present at the location of the crack before the crack is introduced. The solution to the total problem is obtained by superimposing the stress state present before the crack is introduced upon the stress state resulting from the solution of the crack problem with the following boundary conditions:

\[
\begin{align*}
\sigma_{xx}|_{x=0} &= -P(y) \quad \text{for} \quad |y| \leq a \\
U|_{x=0} &= 0 \quad \text{for} \quad |y| > a \\
\tau_{xy}|_{x=0} &= 0 \quad \text{for all} \quad x = 0 \\
\sigma_{ij} &\to 0 \quad \text{as} \quad x \to \infty
\end{align*}
\]

On an intuitive basis, a further condition at the closed end of the crack is

\[
K_I = \text{Limit}_{y \to a} \left(2\pi(y-a)^{1/2} \sigma_{xx}|_{x=0} \right) = 0.
\]

This supposition is easily verified by considering the alternate possibilities: First, suppose that \(K_I < 0\); then calculation of the crack opening displacement shows that the crack surface has deflected through itself, which is impossible. On the other hand, suppose that \(K_I > 0\); then from the definition of stress intensity factor the stress \(\sigma_{xx} > 0\) at \(x=0\) and \(y=a\), which is also impossible because the physical crack extends past \(y=a\) and cannot support a tension stress. The only remaining alternative is that \(K_I = 0\).

For the case of plain strain the Navier equations take the form

\[
\nabla^2 U + \frac{1}{(1-2v)} \frac{\partial}{\partial x} \left( \frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} \right) = 0
\]

\[
\nabla^2 V + \frac{1}{(1-2v)} \frac{\partial}{\partial y} \left( \frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} \right) = 0
\]

where \(U\) and \(V\) are displacement components in the \(x\)- and \(y\)-directions respectively and \(v\) is Poisson's ratio. For the class of problems in which the shearing stresses vanish at all points on the plane \(x=0\) the following potential formulation satisfies Navier's equations.

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