A combined integrating- and differentiating-matrix formulation for boundary-value problems on rectangular domains *

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(Received March 19, 1986 and in revised form May 6, 1986)

Summary.

Integrating and differentiating matrices allow the numerical integration and differentiation of functions whose values are known at points of a discrete grid. Previous derivations of these matrices have been restricted to one-dimensional grids or to rectangular grids with uniform spacing in at least one direction. The present work develops integrating and differentiating matrices for rectangular grids with non-uniform spacing in both directions. The use of these matrices as operators to reformulate boundary-value problems on rectangular domains as matrix problems for a finite-dimensional solution vector is considered. The method requires non-uniform grids which include "near-boundary" points. An eigenvalue problem for the transverse vibrations of a simply-supported rectangular plate is solved to illustrate the method.

1. Introduction

Rotating-beam configurations have traditionally been used to study the vibrations and aeroelastic stability of rotating structures such as helicopter rotor blades and propeller blades. More recently, models involving elastic plates have been proposed to include the effects of spanwise variations in material properties. The fourth-order boundary-value problems associated with both the beam and plate models to not, in general, have useful closed-form solutions. Consequently, most theoretical work on these problems has been asymptotic or numerical in nature.

In one approach to the numerical solution of these problems, harmonic time dependence is assumed to reduce the governing partial differential equation to a differential equation in space variables which includes an eigenvalue. For beam problems, this is an ordinary differential equation. The fundamental derivative which represents beam curvature may be taken as a new dependent variable, and the eigenvalue problem for the beam can be reformulated as an integro-differential equation [3,6,9]. This equation may be conveniently expressed using integral, differential, and boundary-evaluation operators. The operator equation for the continuous solution may further be converted to a matrix operator equation for a finite-dimensional solution vector by evaluating the continuous equation at a finite set of discrete grid points which span the interval of interest. A key question is now the manner in which the matrix operators are approximated.

* Research was supported by the National Aeronautics and Space Administration under NASA Contract Nos. NASI-17070 and NASI-18107 while the author was in residence at ICASE, NASA Langley Research Center, Hampton, VA 23665-5225.
For beam problems, one method for approximating the integral and differential operators involves the use of integrating matrices \([1,4,8]\) and differentiating matrices \([2,7]\). In the simplest terms, these matrices provide, respectively, a means of numerically integrating and differentiating a function whose values are known at a finite set of discrete grid points. A key property of both integrating and differentiating matrices is that their derivation requires only knowledge of the grid points, and no information is needed about the function to be numerically integrated or differentiated. In the case of a beam problem with its single space variable, this property allows the integrating and differentiating matrices based on one-dimensional grids to be used directly as approximations for the integral and differential operators in the matrix operator form of the eigenvalue problem. The result of this approximation is a straightforward matrix eigenvalue problem which can be solved by standard methods. This approach has proved capable of efficiently handling a wide variety of beam problems including beams with concentrated masses, follower forces, and point loadings \([6]\).

For vibration and buckling problems which involve two-dimensional elastic plates, removal of the time dependence from the original boundary-value problem yields an eigenvalue problem which continues to be governed by a partial differential equation. By analogy with the one-dimensional case, it would seem desirable to reformulate this eigenvalue problem as a matrix integro-partial differential equation for a finite-dimensional solution vector on a two-dimensional grid of discrete points. Integrating and differentiating matrices based on two-dimensional grids could then be used to approximate the respective operators resulting, again, in a standard matrix eigenvalue problem.

The present work will explore the potential of this approach by considering an eigenvalue problem associated with the transverse vibration of a simply-supported rectangular plate. This problem consists of the biharmonic eigenvalue equation

\[
\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)^2 u(x, y) = \lambda^2 u(x, y)
\]

(1.1)

for \(0 \leq x \leq A, 0 \leq y \leq B\), and the boundary conditions

\[
\begin{align*}
    u(0, y) &= u_{xx}(0, y) = 0, & u(A, y) &= u_{xx}(A, y) = 0, \\
    u(x, 0) &= u_{yy}(x, 0) = 0, & u(x, B) &= u_{yy}(x, B) = 0.
\end{align*}
\]

(1.2)

In Section 2, equation (1.1) will be reformulated as an integro-partial differential equation consistent with the form of the boundary conditions (1.2). Because (1.2) involves conditions on \(u(x, y)\) itself at all four boundaries, the present approach retains \(u\) itself as the dependent variable. Conversion to a matrix eigenvalue problem will now require the derivation of appropriate integrating and differentiating matrices based on a two-dimensional rectangular grid of discrete points.

One type of integrating matrix for a function of two variables has been previously derived \([5]\). This matrix may be used in two-dimensional problems whose reformulation is possible using an integrating matrix alone, e.g., the plate analogue of a beam with cantilevered boundary conditions. Unfortunately, this matrix is not suitable for the present purposes as its derivation requires that the spacing of the grid points be uniform in at least one direction. The boundary conditions (1.2) lead to a reformulation which will require the use of differentiating matrices to approximate partial derivatives with respect