Symmetry conditions for third order elastic moduli and implications in nonlinear wave theory

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Abstract. Several results are presented concerning symmetry properties of the tensor of third order elastic moduli. It is proven that a set of conditions upon the components of the modulus tensor are both necessary and sufficient for a given direction to be normal to a plane of material symmetry. This leads to a systematic procedure by which the underlying symmetry of a material can be calculated from the 56 third order moduli. One implication of the symmetry conditions is that the nonlinearity parameter governing the evolution of acceleration waves and nonlinear wave phenomena is identically zero for all transverse waves associated with a plane of material symmetry.

1. Introduction

The types of symmetry possessed by elastic materials can be usefully described in terms of the underlying planes of material symmetry. The idea of symmetry planes, as defined by Spencer [1] for example, has been adopted by Cowin and Mehrabadi [2] in their categorization of the elastic symmetries known to exist (the completeness of these symmetry classes was recently proved by Huo and Del Piero [3]). For instance, materials of orthotropic symmetry possess three orthogonal planes of symmetry. In general, the material symmetry is completely specified by the underlying symmetry planes, and vice versa.

The basic result of Cowin and Mehrabadi [2] is a set of conditions on the components of the elastic modulus tensor that are necessary and sufficient that a given direction be normal to a plane of symmetry. An analogous set of conditions for the components of the third order elastic modulus tensor are derived and discussed in Section 2. A simpler set of conditions are also given which are necessary but not sufficient, and are of practical use in determining the underlying material symmetry of a given set of 56 third order elastic constants.

The main result of the paper is derived in Section 4 and concerns the acoustic nonlinearity parameter $\beta$ defined in Section 3. This is the parameter which governs the evolution of acceleration waves in elastic solids, and its
value determines whether or not a given initial disturbance develops into a shock. It is well known [4, 5, 6] that a set of infinitely many transverse wave modes is associated with every plane of material symmetry. By using one of the set of conditions derived in Section 2, it is shown that $\beta$ is identically zero for this class of transverse waves, generalizing a previously known result of Green [7] that $\beta = 0$ in isotropic solids.

There is some ambiguity in the terminology used to discuss elastic coefficients. One approach is to view the moduli as the coefficients in an expansion of stress in terms of strain [8]. The first coefficient, which provides a linear stress-strain relation, is known as the tensor of first order moduli, and the coefficient of the term quadratic in strain is the tensor of second order moduli, etc. Alternatively, one can expand the energy as a power series in strain for materials possessing a strain energy function. Assuming the strain energy is zero for zero strain, i.e. the reference state is taken as the undeformed configuration, then the term linear in strain in the expansion is identically zero. The first non-zero contribution is quadratic in strain and the corresponding coefficient is known as the tensor of second order moduli, even though these moduli are closely related to the first order moduli of the previous definition. In fact they are identical if the constitutive relation expresses the second Piola-Kirchhoff stress tensor as a function of the Green-St. Venant strain tensor [8] and the material is hyperelastic. The term in the energy expansion that is cubic in the strain defines the so-called third order moduli, which are closely related to the second order moduli of the stress-strain definition.

The definitions of second and third order moduli used in this paper are those which follow from the expansion of the strain energy, and they will be discussed explicitly in Section 3. The precise origin of the moduli is immaterial to the discussions of Section 2, where the only relevant attribute used is the symmetry of the components.

A remark on notation: for the remainder of the paper all vectors are unit vectors in three dimensional space; lower case subscripts assume the values 1, 2 and 3; upper case subscripts the values 1, 2, . . . , 6; and the summation convention on repeated subscripts is taken for granted.

2. Necessary and sufficient conditions for the existence of a plane of symmetry

Given an elastic stiffness or compliance tensor, it is not immediately clear what symmetries, if any, the corresponding material possesses. This question of determining the symmetry was addressed and answered by Cowin and Mehrabadi [2] for the moduli of linear elasticity, i.e. the second order moduli. The components of the modulus tensor are $C_{ijkl}$ relative to a rectangular basis,