On Saint-Venant's problem for elastic dielectrics

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Abstract. The equilibrium theory of linear piezoelectricity is considered. Saint-Venant's problem for a homogeneous and anisotropic piezoelectric cylinder is studied.

1. Introduction

The importance of Saint-Venant's celebrated memoirs [1, 2] on what has long since become known as Saint-Venant's problem, requires no emphasis. Indeed, a comprehensive bibliography of the vast and varied literature to which the work contained in [1, 2] has given impetus would multiply the length of this study. We recall that Saint-Venant's problem consists in determining the equilibrium of an elastic cylinder, loaded only by surface forces distributed over its plane ends. Saint-Venant's approach to the problem is based on a relaxed statement in which the pointwise assignment of the terminal tractions is replaced by prescribing the corresponding resultant force and resultant moment. In [3], we have established a simple method of deriving Saint-Venant's solution. The advantage of this method is that it does not involve artificial a priori assumptions and offers a rational scheme to obtain Saint-Venant's solution.

In this paper an adaptation of the method of [3] is used to study the relaxed Saint-Venant's problem within the linear theory of elastic dielectrics.

2. Preliminaries

Throughout this paper $\mathcal{B}$ denotes the interior of a right cylinder of length $h$ with the open cross-section $\Sigma$ and the the lateral boundary $\Pi$. We assume that the generic cross-section $\Sigma$ is a simply connected region and denote by $\Gamma$ the boundary of $\Sigma$. We let $\bar{\mathcal{B}}$ denote the closure of $\mathcal{B}$, call $\partial \mathcal{B}$ the boundary of $\mathcal{B}$, and designate by $n$ the outward unit normal of $\partial \mathcal{B}$. Letters in boldface stand for tensors of an order $p \geq 1$, and if $v$ has the order $p$, we write $v_{ij\ldots s}$ ($p$ subscripts) for the components of $v$ in the underlying rectangular
Cartesian coordinate frame. We shall employ the usual summation and differentiation conventions: Greek subscripts are understood to range over the integers (1, 2), whereas Latin subscripts — unless otherwise specified — are confined to the range (1, 2, 3); summation over repeated subscripts is implied and subscripts preceded by a comma denote partial differentiation with respect to the corresponding coordinate. The rectangular Cartesian coordinate frame is supposed to be chosen in such a way that the \( x_3 \)-axis is parallel to the generators of \( B \), and the \( x_1 O x_2 \)-plane contains one of the terminal cross-sections, while the other is in the plane \( x_3 = h \). We denote by \( \Sigma_1 \) and \( \Sigma_2 \), respectively, the cross-sections located at \( x_3 = 0 \) and \( x_3 = h \).

We consider the equilibrium theory of linear piezoelectricity (see, for example, [4–6]). Let \( u \) denote the mechanical displacement field, and let \( \varphi \) denote the electric potential. We denote by \( u \) the four-dimensional vector field on \( B \), defined by \( u = (u, \varphi) \equiv (u_1, u_2, u_3, \varphi) \). The electric enthalpy corresponding to \( u \) is

\[
U(u) = \frac{1}{2} C_{ijrs} u_{i,j} u_{r,s} + \epsilon_{rij} \varphi_{,j} u_{i,j} - \frac{1}{2} \epsilon_{ij} \varphi_{,i} \varphi_{,j},
\]

(2.1)

where \( C_{ijrs} \), \( \epsilon_{mj} \) and \( \epsilon_{rs} \) are constitutive coefficients which satisfy the symmetry relations

\[
C_{ijrs} = C_{rsij}, \quad \epsilon_{mj} = \epsilon_{mj}, \quad \epsilon_{ij} = \epsilon_{ij}.
\]

(2.2)

We assume that the material is homogeneous so that the constitutive coefficient are constants. The constitutive equations for an anisotropic body are given by

\[
t_{ij}(u) = C_{ijrs} u_{r,s} + \epsilon_{kij} \varphi_{,k},
\]

\[
D_i(u) = \epsilon_{ijk} u_{j,k} - \epsilon_{ij} \varphi_{,j},
\]

(2.3)

where \( t(u) \) is the stress field and \( D(u) \) is the electric displacement field, associated with \( u \).

We suppose that the electric enthalpy is a definite quadratic form in the components of the strain tensor and electric field. It follows that \( U(u) = 0 \) if and only if \( u_i = a_i + \epsilon_{nr} b_n x_n \), \( \varphi = c \), where \( a_i \), \( b_i \), \( c \) are arbitrary constants and \( \epsilon_{nr} \) is the alternating symbol. Let

\[
H = \{ u^0; u^0 = (u_i^0, \varphi^0), u_i^0 = a_i + \epsilon_{nr} b_n x_n, \quad \varphi^0 = c \},
\]

(2.4)

where \( a_i \), \( b_i \) and \( c \) are constants. If \( u \in H \), then \( u \) is called a rigid motion.