The representation theorem for isotropic, linear asymmetric stress-strain relations

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In 1974, Gurtin [1] gave an elegant proof of the representation theorem for isotropic, linear stress-strain relations, considerably improving the one supplied by the same author in [2]. We cite this theorem literally as follows (notations will be explained subsequently):

Representation Theorem. Let $S: \text{Sym} \to \text{Sym}$ be linear and isotropic. Then there exist scalars $\lambda$ and $\mu$ such that

$$S(E) = \lambda (\text{tr } E) I + 2\mu E \quad (\forall E \in \text{Sym}).$$

In recent years, micropolar elasticity has been intensively investigated (see, for example, [3]), where the stress and strain are no longer symmetric. Therefore, an extension of the representation theorem to asymmetric relations would appear to be of some interest. The aim of the present brief note is to offer a proof of such an extended theorem.

Let $\mathbb{R}$ be the space of scalars and $\mathbb{V}$ a 3-dimensional Euclidean space with scalar product $uv$ and vector product $u \wedge v$. $\text{Lin}$ is the space of all linear transformations (or simply tensors) from $\mathbb{V}$ into itself, with the unit tensor $I$. $\text{Sym}$, $\text{Skw}$ and $\text{Orth}$ denote, respectively, the sets of symmetric, skew-symmetric and orthogonal elements in $\text{Lin}$. Given a tensor $T$, $T^*$ denotes its transpose and $\text{tr } T$ its trace. The transpose of the tensor product $u \otimes v$ is

$$(u \otimes v)^* = v \otimes u.$$ (2)

Every tensor $T$ possesses at least a right proper direction $r (\neq 0)$,

$$Tr = \lambda r,$$ (3)

and a left proper direction $l (\neq 0)$,

$$lT = \lambda l,$$ (4)

$\lambda$ being the associated proper value. When $r = l$, we call $r$ simply a proper direction. Every $Q \in \text{Orth}$ possesses (modulo multiplicative constants) only one proper direction $r$:

$$Qr = rQ = (\det Q)r.$$ (5)

Given non-coplanar $u, v, w \in \mathbb{V}$, any $A \in \text{Skw}$ may be expressed as

$$A = \xi (u \otimes v - v \otimes u) + \eta (v \otimes w - w \otimes v) = \xi (w \otimes u - u \otimes w),$$ (6)
where $\xi$, $\eta$, $\zeta \in \mathbb{R}$. Every skew-symmetric tensor $A$ possesses also only one proper direction, the associated proper value being zero. The proper direction of the skew-symmetric tensor $u \otimes v - v \otimes u$ is $u \wedge v$, because
\[
(u \otimes v - v \otimes u)(u \wedge v) = (u \wedge v)(u \otimes v - v \otimes u) = 0.
\]

**Definition.** A function $F: \text{Lin} \rightarrow \text{Lin}$ is isotropic if and only if
\[
QF(T)Q^* = F(QTQ^*) \quad (\forall T \in \text{Lin}, \ Q \in \text{Orth}). \tag{7}
\]

**Lemma.** Let $F: \text{Lin} \rightarrow \text{Lin}$ be linear and isotropic. Then there exists a scalar $\alpha$ such that
\[
F(u \otimes v - v \otimes u) = 2\alpha(u \otimes v - v \otimes u) \quad (\forall u, v \in \mathcal{V}). \tag{8}
\]

**Proof.** Firstly we consider the case of non-collinear $u, v \in \mathcal{V}$. For brevity, we denote the skew-symmetric argument by
\[
\mathcal{C} = u \otimes v - v \otimes u \tag{9}
\]
and its proper vectors by
\[
\mathcal{w} = u \wedge v, \quad r = w/|w|, \tag{10}
\]
then
\[
\mathcal{C}r = r\mathcal{C} = 0. \tag{11}
\]
Introducing the orthogonal tensor
\[
R = -I + 2r \otimes r, \tag{12}
\]
which is a rotation about $r$ though $\pi$, we have
\[
R\mathcal{C}R^* = (-I + 2r \otimes r)\mathcal{C}(-I + 2r \otimes r) = \mathcal{C} - 2Cr \otimes r - 2r \otimes r\mathcal{C} + 4r \otimes r\mathcal{C} \otimes r = \mathcal{C}, \tag{13}
\]
where the Eqns. (11) have been used. For this particular orthogonal tensor $R$, the isotropy equation (7) becomes
\[
RF(\mathcal{C})R^* = F(\mathcal{C}) \quad \text{or} \quad RF(\mathcal{C}) = F(\mathcal{C})R, \tag{14}
\]
and, consequently,
\[
R[F(C)w] = F(C)Rw = F(C)w, \tag{15}
\]
\[
[wF(C)]R = wRF(C) = wF(C). \tag{16}
\]
Thus $F(C)w$ and $wF(C)$ present the right and left proper directions of $R$. Since $R$ has only one proper direction $w$, we can write
\[
F(C)w = \xi w \quad \text{and} \quad wF(C) = \xi w. \tag{17}
\]
Referred to $u, v, w$ as basis vectors, $F(C)$ may be expressed in the dyadic form:
\[
F(C) = \xi_{11}u \otimes u + \xi_{12}u \otimes v + \xi_{13}u \otimes w + \xi_{21}v \otimes u + \xi_{22}v \otimes v + \xi_{23}v \otimes w + \xi_{31}w \otimes u + \xi_{32}w \otimes v + \xi_{33}w \otimes w. \tag{18}
\]
Generally, the coefficients $\xi_{ij}$ $(i, j = 1, 2, 3)$ depend on $u$ and $v$. Inserting (18) into (17),