The Thermoelastic Material-momentum Equation*

C. DASCALU and G.A. MAUGIN
Laboratoire de Modélisation en Mécanique, Université Pierre et Marie Curie, URA-CNRS 229, b.p. 162, Tour 66, 4, place Jussieu, 75252 Paris Cedex 05, France

Received 19 April 1994; in revised form 2 March 1995

Abstract. The equation of material momentum, or pseudomomentum, is obtained for thermoelastic materials. This is done in the classic theory, based on the heat conduction hypothesis, and also in the framework of a thermoelasticity approach involving no dissipation of energy, as recently proposed by Green and Naghdi. The results are applied to the thermoelastic fracture problem. When the pseudomomentum equation is written in global form, for a fractured body, it provides path-domain invariant expressions for the thermoelastic energy-release rate.

1. Introduction

The equation of material momentum, or as we prefer to say following physicists, pseudomomentum [1, 2], was introduced in its static form in elasticity by Eshelby [3] who first recognized its essential capacity to provide the configurational forces (also called material forces) acting on defects in extension through elastic bodies, when it is written in global form. The dynamics was further incorporated in this equation by Eshelby himself [4] and others [5, 6], the most recent formulation being that in Maugin and Trimarco [7] and Dascalu and Maugin [8] – see also the synthesis view of material forces presented in Maugin [9]. In quasistatic elastic fracture, the equation of pseudomomentum leads to the celebrated J-integral of Rice [10] as an equivalent expression for the energy-release rate.

Although the ‘elastic’ pseudomomentum equation has been known for some time, and it was further generalized so as to include additional effects such as plasticity, viscoelasticity and electromagnetism [9], its generalization to thermoelasticity is still a debated question. It is the aim of this paper to bring some clarification to this particular application. There are two attempts at obtaining such an equation in thermoelasticity. One [11] is unfortunately based on the misconception that the balance of pseudomomentum is necessarily of variational-Noetherian origin while it in fact is a general law of physics. As thermoelasticity with heat conduction is not physically derivable from such a variational principle, only abstract ad-hoc manipulations allow these authors to obtain an equation with the appearance of pseudomomentum qualification including non-local terms in time. In the subsequent application to fracture problems [12], a similar, but formal, notion of energy-release rate was needed for the relation with their material force.

* Dedicated to the memory of Paul M. Naghdi.
The second approach to the thermoelastic balance of pseudomomentum is due to Epstein [13] and Epstein and Maugin [14] who, first in quasistatics, and then in dynamics, identified an expression of the thermoelastic material force (see below) in terms of entropy and temperature gradient, and provided a geometrical setting to finite-strain thermoelasticity viewed as a quasi-plastic process. On the basis of these preliminary results we formulate here an equation of pseudomomentum which, when applied to fracture, indeed provides an equivalent expression of the (classic) energy-release rate but in thermoelasticity (compare to [15, 16, 17]).

Furthermore, Green and Naghdi [18] have recently presented a formulation of the equations of thermoelasticity that does not involve energy dissipation. An important step in the construction of their approach is the introduction of the notion of 'thermal displacement' (in fact an old notion as we shall see), which is the analog for thermal fields of the mechanical displacement for mechanical ones. This variable does play an important role in our formulation of the equation of pseudomomentum. The later is given in general form so that it contains, as particular cases, the case of the classical theory of thermoelasticity and also the Green–Naghdi description.

When the global balance of pseudomomentum is stated for a cracked body, path-domain invariant integral expressions are obtained for the thermoelastic energy-release rate. The corresponding results of pure elasticity are recovered when thermal effects are ignored. We also notice that the local form of the pseudomomentum equation is not simply the convection of physical momentum back to the reference configuration as is the case in pure elasticity, but it also accounts for the entropy equation. This contribution of all 'degrees of freedom' which present material gradients, was recognized earlier in rather complex theories of continua (e.g. magnetoelectromagnetism in deformable materials, Maugin [6], magnetoelasticity of ferromagnets [19], and liquid crystals [20]). This ultimately follows from the fact that the equation of pseudomomentum represents the invariance – or lack of invariance in the presence of material inhomogeneities – of the whole theory, under material coordinates.

The paper is structured as follows. In Section 2 the equation of pseudomomentum is deduced in the framework of the classical theory of thermoelasticity, while in Section 3 the corresponding equation is given for the theory without dissipation. These equations are rewritten in global form for a cracked body in Section 4, providing thus the expressions of material forces that act on the progressing crack-tip. It is indeed proven that they are related to the rate of energy released during thermoelastic fracture. Finally, integral-invariant expressions of these energy-configurational forces are established in the framework of each theory. As a rule we use the notation of previous works (e.g. Dascalu and Maugin [8], Maugin [9]).