Stress intensity factors in yielding materials by the method of caustics

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ABSTRACT
The method of caustics, as it has been developed by the author, was applied up to now to evaluate the stress-intensity factors in cracked plates under conditions of generalized elastic plane stress [1]. According to this method the partially reflected light beam either on the front or on the rear face of a polished cracked plate was deviated at the vicinity of the crack-tip because of the large constraint at this area, due to the Poisson effect and created a caustic. It was shown that the dimensions of the caustic are directly and intimately related to the values of the stress-intensity factor components. In this paper the theory developed for the elastic case of loading was extended to incorporate the case of a yielding material at the vicinity of the crack-tip, which can be represented by the Dugdale–Barenblatt cohesive force model, and presents a stress-strain curve resembling an elastic perfectly-plastic material. The equations for the initial curve engendering the caustic and the generalized epicycloid, which represents the caustic, were established. It was shown that at the early stage of yielding, at the vicinity of the crack-tip, the typical shape of a quasi-circular caustic appears, which later on, in a higher step of loading, evolutes to the well-known shape of the plastic wedge-shaped zone. Experimental evidence with steel and plastic specimens corroborated the theoretical results.

1. Introduction
The evaluation of the opening-mode stress intensity factor in cracks existing in plates of elastic-plastic materials was not adequately treated, as it was desirable for application in fracture mechanics. Considerable progress, however, has been made with the introduction of a simple model for plane stress yielding proposed by Dugdale [1] and a similar one, postulated by Barenblatt [2]. According to this model yielding is assumed to be confined to a narrow zone constituting a protrusion of the crack ahead of the crack-tip. The model was analysed by considering the effect of yielding as making the crack longer by an amount equal to the plastic zone size R, with cohesive stresses in the plastic zone, which act on the extended crack surface in such a manner so that they restrict the opening of the crack-lips. Since the applied stresses on the plate and the restraining cohesive stresses create singularities at the outer tip of the plastic zone, this zone can be defined by the fact that these singularities, which are of opposite sign, cancel each other and therefore bounded stresses exist at the outer tip.
A further simplification of the Dugdale–Barenblatt model is to accept that the restraining cohesive stress distribution along the plastic zone R is constant and equal to the yield stress $\sigma_0$ for the perfectly plastic material. Rosenfield, Dai and Hahn [3] have shown that this type of model is particularly relevant to fully developed plane stress yielding in generalized plane-stress problems, by revealing plastic zones by the etch-pitting technique. Plastic flow revealed by this technique consists of two intersecting shear bands through the thickness of the sheet at angles of 45 deg., and yielding is localized to a narrow zone of height approximately equal to the thickness of the plate. With the above simplifications it is easy to extend the solution for the elastic crack problem, based on Muskhelisvili’s method of complex stress functions [4] or Westergaard’s similar method [5, 6], to the solution of the corresponding problem for fully developed plane stress yielding in thin cracked sheets.
The Dugdale–Barenblatt model has been also analyzed for various other stress configurations. Bilby and Swinden [7] and Smith [8] considered the case of an infinite periodic array of collinear cracks, while Bilby, Cottrell and Swinden [9], Smith [10], Goodier and Field [11], Rice [12], and others, examined similar problems for this model, as well as cases of cracked
plates in an edge-sliding mode and antiplane shear mode of deformation. Finally, a very
comprehensive and thorough study of the general problem of stress distribution around cracks
is given by Rice [13], where the reader may be addressed for a concise consideration of the
mathematical analysis in the mechanics of fracture.

On the other hand, the experimental revelation of the shape of plastic zones by Hahn and
co-workers [14] and by others was restricted mainly to an advanced loading step of the cracked
plate, where deep wedge-shaped plastic zones appear in the specimen, which either are spread
out in approximately normal directions to the crack-axis (plastic-hinge type), or they are pro-
ject ed forward as large tapering wedges (plastic-wedge type). Other experimental methods
based on interferometry [23, 24] and holographic interferometry [25] are applicable only at the
preliminary stage of small scale yielding.

In this paper the optical method of caustics, which was previously applied to elastic brittle
fracture problems [17], was extended to perfectly plastic solutions based on the Dugdale–
Barenblatt model as applied to metals and alloys. In this way this method may cover the whole
range of deformation of the cracked plate.

2. Complex Stress Functions for the D–B Model

Consider a thin plate made of an elastic-perfectly plastic material, which contains either an
edge or an internal transverse crack of length 2a, and submitted to a uniform stress field at
infinity represented by the normal stresses $\sigma_{ox}$ and $\sigma_{oy}$. A system of Cartesian coordinates $Oxy$
related to the crack so that the Ox-axis coincides with the crack-axis, while the Oy-axis
passes through the center of the crack (Fig. 1).

![Figure 1. Model of Dugdale-Barenblatt crack for an elastic-perfectly plastic material.](image)

If the loading of the material is such to create small-scale yielding, this yielding is assumed
to be confined to a narrow zone along the x-axis. The D–B model is analyzed by regarding the
effect of yielding as extending the crack-length by the length $R$ of the plastic zone. Along the
extended length of the crack ($a < x \leq (a + R)$) cohesive stresses are acting to restrain the opening
of the crack. The applied load and the restraining stresses are creating inverse square root
singularities at the outer tip of the plastic zone. These singularities, which are of opposite
sign, must cancel out and therefore yield bounded stresses at the outer tip of the extended