Power-law creep of a material being compressed between parallel plates: a singular perturbation problem

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(Received July 1, 1983 and in revised form December 12, 1983)

Summary

The flow of a power-law creeping or viscous material which is being compressed in a narrow gap between parallel plates is studied. A perturbation scheme based on the small gap size is developed and the approximations which lead to classical lubrication theory are formally identified. The solution obtained from lubrication theory is shown to correspond to an outer solution which is not uniformly valid because it predicts infinite longitudinal stresses along both the centerline of the gap and across the entire gap on the line connecting the plate midpoints. The failure of lubrication theory to describe the flow in these regions is not due to an inherent failure of the power-law constitutive equation to model material behavior, but is due to a breakdown in the approximations made in lubrication theory. The solution is corrected by constructing inner solutions in the regions where lubrication theory fails and a uniformly valid solution for the stress field and velocity field is obtained.

1. Introduction

The present study is a detailed analysis of the flow of a power-law creeping material which is being squeezed between parallel plates. Scott [1] was the first to examine this squeezing-flow problem. He was interested in deducing material properties from readings obtained from a parallel-plate plastimeter. A summary of Scott's analysis may be found in many papers; for example, Leider and Bird [2], and Brindley, Davies and Walter [3]. Previous analyses of this problem have used the approximations of classical lubrication theory from the outset, based on heuristic arguments concerning the smallness of the gap between the plates and neglecting inertial effects. Furthermore, in the majority of the work the goal has generally been to obtain rather global results, such as an expression for the net force on the plates in terms of the plate velocity and material properties. Here we present a detailed analysis of the flow by formally developing the solution in terms of a perturbation expansion based on the small gap size. This formal treatment reveals that, in the case of a power-law fluid, the lubrication approximation fails to be uniformly valid and inner solutions are required near the gap centerline and near the line connecting the plate midpoints. Flow details in these regions are relevant to the micromechanical analysis of void growth and suspended particle transport.

For the power-law creeping material considered here the effective viscosity is proportional to the inverse of a power of the second stress invariant. In the lubrication theory the second stress invariant is approximated by the square of the shear stress. Consequently, along the centerline of the gap, where the shear stress vanishes by symmetry, the effective
viscosity predicted by the lubrication approximation is infinite. Therefore, since the strain rate is nonzero there, the lubrication theory predicts infinite stress at the centerline of the gap. This physically unacceptable singularity is not inherent in the constitutive equation for a power-law creeping material but is simply a failure of the lubrication approximation. In particular, the lubrication approximation is not uniformly valid and an inner solution must be constructed in the neighborhood of the centerline.

For similar reasons the lubrication approximation also fails across the entire gap in the vicinity of the line which connects the plate midpoints. In this case the failure is not only due to the fact that the shear stress vanishes (by symmetry), but is also due to a breakdown in the scaling for the velocity field. As we shall see, the lubrication approximation assumes that the component of velocity parallel to the plates is large compared to the perpendicular component. This assumption is incorrect in the plate midpoint region, since the velocity components are expected to be of the same order of magnitude there.

We shall find that the inner solutions discussed above are necessary in order to obtain a uniformly valid description of the stress field, but they do not significantly affect the velocity field predicted by the outer or lubrication-theory solution.

The goal of the present paper is primarily aimed at resolving a somewhat mathematical question. Namely, we determine the precise structure of the solution for the compressive flow of a power-law viscous material and demonstrate that a solution free from any undesirable singular behavior may be obtained by using the method of matched asymptotic expansions. Nevertheless, some discussion of the practical value of the present problem is appropriate here. This requires an examination of the usefulness and limitations of the constitutive equation being considered.

The materials for which the power-law model is useful include metals at high temperature (generally temperatures greater than half of the melting temperature) and some polymeric liquids. At temperatures above about half the melting temperature and strain rates greater than $10^{-6}$ sec$^{-1}$ many metals are modeled reasonably well by the power-law equation considered here [4,5,6]. The deformation mechanism in this case is generally referred to as dislocation creep. For strain rates below $10^{-6}$ sec$^{-1}$, however, diffusional creep dominates and the material behaves in a linearly viscous fashion. Due to this linearly viscous behavior at small strain rates the power-law constitutive equation becomes a poor model of the material in the same regions where the lubrication approximation fails, i.e., in regions where the shear stress or shear strain rate becomes small. However, the constitutive equation generally fails at such small values of the strain rate that the size of the inner regions which correct the lubrication theory can often be much larger than the size of the regions where the power-law equation fails. In particular, in the present analysis we find that we enter the inner regions when the shear strain rate is of the order $V/h$ where $V$ is the plate velocity and $h$ is the gap width. Consequently, whenever this quantity is greater than $10^{-6}$ sec$^{-1}$, (i.e., the strain rate below which we typically expect linearly viscous behavior) the present analysis is relevant since the regions where the constitutive equation fails would be small and embedded within the inner regions which are examined here. We should note that the linearly viscous behavior at very small stresses could be modeled using a generalized constitutive equation which includes an additive constant in the effective viscosity, however, such a generalization here only complicates the analysis and does not serve our primary purpose.

Similarly, in the case of polymeric liquids a power-law model may also be an adequate description. However, although polymeric liquids are generally shear thinning many polymeric liquids show tensile thickening behavior which is in contrast to the tensile