Distributions of sources and normal dipoles over a quadrilateral panel

J.N. Newman
Department of Ocean Engineering, Massachusetts Institute of Technology, Cambridge, MA 02139, USA

(Received May 2, 1985 and in final form September 3, 1985)

Summary

The potential due to a distribution of sources or normal dipoles on a flat quadrilateral panel is evaluated for the cases where the density of the singularities is constant, linear, bilinear, or of arbitrary polynomial form. The results in the first two cases are consistent with those derived previously, but the present derivation is considered to be simplified. In particular, the constant dipole distribution is derived from a geometric argument which avoids direct integration; this derivation applies more generally on a curvilinear panel bounded by straight edges.

Also presented are multipole expansions for the same potentials, suitable for use when the distance to the field point is substantially larger than the panel dimensions. Algorithms are derived to evaluate the coefficients in these expansions to an arbitrary order.

1. Introduction

A wide variety of practical problems in hydro- and aerodynamics may be solved using boundary-integral methods, with the velocity potential constructed from distributions of sources and normal dipoles on each panel of the discretized boundary surface. If the strength of the singularities is assumed constant on each panel, and the boundary-integral equation is solved by collocation, a linear system of algebraic equations results with canonical matrix elements given by unit-strength distributions on each panel. These must be evaluated at appropriate nodal points on every panel. With N panels used to describe the body surface, there are \( N \times N \) matrix elements for both the source and normal dipole. Typical three-dimensional solutions require on the order of 1000 panels, with \( O(10^6) \) matrix elements to be evaluated. The computation of these elements is an important task in the numerical solution.

Hess and Smith [1–3] introduced this general technique, using constant-strength distributions on quadrilateral flat panels, and derived closed-form expressions from the matrix elements by evaluating analytically the surface integrals over each panel. One concept stressed in the first two references is that a surface integral over the quadrilateral panel (or more generally over a polygonal panel with an arbitrary number of sides) can be expressed as a superposition of integrals over a set of infinite parallel strips. Each strip is defined by one side of the panel, and the value of the corresponding integral depends only on the coordinates of that side. This decomposition is effective in the algorithmic sense, since the computations can be performed in sequence for each side. In a simpler and more
direct approach followed by Hess and Smith [3], the surface integral for the source
distribution is reduced to a line integral around the perimeter of the panel, with the same
implication that the contribution from each side can be treated independently.

In subsequent extensions linear distributions have been used on triangular panels to
provide a more continuous description of the solution. The appropriate matrix elements
for these distributions have been derived by Yeung [4] and Webster [5]. However the
fundamental connections which exist between their results and those obtained by Hess
and Smith do not appear to be appreciated.

A more unified derivation including and extending these earlier analyses is the
objective of the present paper. First, it will be shown that the potential due to a normal
dipole distribution with constant moment can be derived from an appropriate sum of
surface integrals over infinite sectors, corresponding to each vertex of the panel and
bounded by semi-infinite extensions of the adjacent sides of the panel. In this way the
dipole potential may be expressed as a sum of terms depending only on the properties of
each vertex, as opposed to each side. The latter feature could be deduced directly from the
result of Hess and Smith, simply by regrouping of the pairs of terms in their equations.
However, the derivation used here employs the Gauss-Bonnet theorem to evaluate the
dipole potential directly, in terms of the included angle of each vertex projected on a
plane normal to the axis between the vertex and the field point. In this manner the
analysis associated with direct integration over the panel surface can be avoided. An
additional feature of the present derivation is that it is valid for an arbitrary curvilinear
panel surface bounded by straight segments.

In view of the equivalence between the constant dipole distribution and a vortex
filament surrounding the panel, this method can also be used in lifting-surface applica-
tions to derive the velocity potential of a vortex lattice.

The corresponding result for a source distribution of constant strength on a flat panel
is obtained by integration from the dipole distribution, and involves only one elementary
integral. Although the approaches differ, this portion of our analysis appears algebraically
similar to that presented by Hess and Smith [3].

For the linear distributions of singularities two canonical integrals are considered for
the source potential, and evaluated with recourse to only one elementary integral. The
corresponding results for the dipole are obtained in a similar manner. The potentials for
singularities of constant strength are utilized to simplify the analysis. The same approach
is used to derive a bilinear source distribution, which provides a representation of
continuous singularity distributions on quadrilateral panels.

A more general recursive scheme is developed in Section 5 to evaluate source and
dipole distributions of arbitrary higher-order polynomial form.

Section 6 is devoted to a far-field multipole approximation of the above results. The
efficiency of such an approach is emphasized by Hess and Smith [1,2] with a point source
and quadrupoles used in place of the constant-strength source distribution when the field
point is sufficiently far from the panel. The corresponding strengths of these point
singularities are the area and second moments, respectively, of the panel. (The dipole
terms proportional to the first moments are eliminated by locating the point singularities
at the centroid of the panel.) We extend this technique to include linear and higher-order
distributions, and we retain additional terms in the multipole expansions proportional to
the third and fourth moments of the panel area. The latter extension increases the
accuracy of the far-field approximation, and extends its domain of application. Al-
gorithms are described for computing arbitrary moments of each panel.