Circles on the slowness surface of a cubic elastic material

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ABSTRACT

It is well-known that either the outer or the medial sheet of the slowness surface of an elastic material with cubic symmetry intersects the cube faces in circles. It is shown here that there exist on the next sheet (medial or outer) three pairs of circles centred on the symmetry axes and situated in planes parallel to the cube faces.

1. Introduction

The slowness surface $S$ of an elastic material with symmetry elements belonging to the cubic system has been described by a number of authors, the most extensive study being due to Miller and Musgrave [1]. We point out in this note a simple property of $S$ which appears to have escaped notice in previous work, namely the presence on the outer or medial sheet of a family of six circles. The circles have no substantive bearing on the behaviour of elastic body waves, but they serve as an aid to the global visualization of $S$ and as a check on numerical calculations of sections of $S$.

In the algebra which follows vector and tensor components relate to an orthonormal basis $\{e_1, e_2, e_3\}$ aligned with the equivalent axes of symmetry of the material. For brevity the term face is used as a synonym for plane of material symmetry.

2. The equation of $S$

For an elastic material possessing cubic symmetry relative to a stress-free configuration $N$ the linear elasticity tensor $C$ has components

$$C_{ijkl} = c_{12}\delta_{il}\delta_{kl} + c_{44}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$$

$$+ (c_{11} - c_{12} - 2c_{44}) \sum_{r=1}^{3} \delta_{ij}\delta_{kr}\delta_{lr}$$

[1], and the requirement that $C$ shall be positive definite (or strongly convex)
imposes on the independent elastic constants $c_{11}$, $c_{12}$, $c_{44}$ the inequalities

$$c_{11} + 2c_{12} > 0, \quad c_{11} - c_{12} > 0, \quad c_{44} > 0. \quad (2)$$

The equation of the slowness surface $S$ is

$$|\det (C_{pqi} s_p s_q - \rho \delta_{iq})| = 0, \quad (3)$$

where $s_i$ are the components of the slowness vector $s$ and $\rho$ is the density of the material in $N$. The result of substituting the expression (1) into equation (3) and expanding the determinant can be written as

$$\Gamma^3 + a (s_1^2 + s_2^2 + s_3^2) \Gamma^2 + (a^2 - b^2)(s_2^2 s_3^2 + s_3^2 s_1^2 + s_1^2 s_2^2) \Gamma + (a-b)^2 (a + 2b)s_1^2 s_2^2 s_3^2 = 0, \quad (4)$$

where

$$\Gamma = c_{44}(s_1^2 + s_2^2 + s_3^2) - \rho, \quad a = c_{11} - c_{44}, \quad b = c_{12} + c_{44}. \quad (5)$$

We denote by $O$ the centre of $S$ and note the implications

$$a + b > 0, \quad a + 2b > 0 \quad (6)$$

of the inequalities (2).

3. Facial circles

The roots of the cubic equation (4) correspond to the three sheets of $S$. One root is zero whenever a component of $s$ vanishes and we deduce from (5) that the associated slowness sheet intersects each of the faces $s_i = 0 \ (i = 1, 2, 3)$ in a circle of radius $(\rho/c_{44})^{1/2}$. For all values of the elastic constants obeying the conditions (2) one sheet of $S$ thus contains three facial circles with centre $O$. The facial circles belong to the outer sheet when $-\frac{1}{2}a < b \leq a$ and to the medial sheet when $-b < a \leq b$.

When $a = b (>0)$ the material under consideration is isotropic. The roots of equation (4) are then $-a(s_1^2 + s_2^2 + s_3^2)$ and 0 (twice), and the sheets of $S$ are spheres, the outer and medial sheets coinciding. A second degenerate case arises when $b = 0$ (and, from (6), $a > 0$). In this instance the roots of equation (4) are $-as_1^2$, $-as_2^2$, $-as_3^2$, and it follows from (5) that the sheets of $S$ are identical oblate spheroids. We assume henceforth that $a \neq b \neq 0$.

4. Non-facial circles

Choosing the base vector $e_3$ as axis we introduce spherical polar slowness components $s$, $\theta$, $\phi$ such that

$$s_1 = s \sin \theta \cos \phi, \quad s_2 = s \sin \theta \sin \phi, \quad s_3 = s \cos \theta.$$