A new notation for stress and strain

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ABSTRACT

This note is concerned with a new notation for the stress and deformation gradients. It is intended to be both clear and elegant.

A new notation for the stress and deformation gradient tensors is presented in this note. It is designed to emphasize the role played by the various material configurations themselves.

It is common to denote the Cartesian coordinates of a particle in the reference configuration by \( X_A \) \((A = 1, 2, 3)\) and those of the same particle in the current configuration by \( x_i \) \((i = 1, 2, 3)\). The deformation gradient is

\[
p_{iA} = \frac{\partial x_i}{\partial X_A},
\]

and we define

\[
J = \det p_{iA}.
\]

If the Cauchy stress in the current configuration is \( \sigma_{ij} \) then the first Piola-Kirchhoff stress is given by

\[
\pi_{Ai} = J \frac{\partial X_A}{\partial x_j} \sigma_{ij},
\]

or, more compactly,

\[
\pi = J p^{-1} \sigma.
\]

We have abbreviated \( \pi_{Ai} \) to \( \pi \) and \( p_{Ai}^{-1} \) to \( p^{-1} \) but there is nothing in the notation of (4) to remind us of the two configurations involved. Therefore, let us write \([A, i]\) instead of \( \pi_{Ai} \) allowing square brackets to denote stress. We may now condense \([A, i]\) to \([1, 2]\), rather than \(\pi\), where the label 1 denotes the material configuration and the label 2 denotes the current configuration. Similarly, we employ round brackets for strain so that \( p_{Ai}^{-1} \) is replaced by \((A, i)\), which in turn is condensed to \((1, 2)\). We also define \( J_{12} \) by

\[
J_{12} = \det (2, 1) = \det p = J.
\]

Equation (3) now becomes

\[
(1, 2) = (2, 1) = J = J_{12}.
\]
\[ [1, 2] = J_{12}(1', 2')[2', 2], \] (6)

where the 2's have been primed to indicate that the relevant suffices are to be summed over. The symbol [2, 2] is the Cauchy stress in the 2 (current) configuration and [1, 1] is the Cauchy stress in the 1 (material) configuration.

There are some chain rules which are easy to derive. If 3 is a third configuration we have
\[ J_{12}J_{23} = J_{13}, \quad J_{12} = J_{21}, \quad J_{11} = 1. \] (7)

Also we have
\[ (1, 2')(2', 3) = (1, 3), \quad (1, 2')(2', 1) = (1, 1), \] (8)

where (1, 1) and (2, 2) etc. each represent the unit tensor. We may also show from (6) that
\[ [1, 3] = J_{12}(1, 2')[2', 3]. \] (9)

This equation relates [1, 3] and [2, 3] without directly bringing in the Cauchy stress [3, 3]. On allowing the configuration 3 to coincide with configuration 2, i.e. \(3 \rightarrow 2\), it becomes clear that (6) is a special case of (9).

In equations (7), (8) and (9), the configurations 1, 2, 3 need not necessarily occur in that order in time.

If we denote by \(\rho_1\) and \(\rho_2\) the densities in the 1 and 2 configurations respectively the equation of mass balance becomes
\[ \rho_1 = J_{12}\rho_2. \] (10)

Let us denote by \(x^{(2)}\) the current coordinates of a particle in the 2 configuration and let \(b(X)\) be the external body force per unit mass. The equations of motion, valid for any configurations 1 and 2, are then
\[ [1', 2], \rho_1 + \rho_1 b = \rho_1 \dot{x}^{(2)} \] (11)

whilst in the previous notation they were
\[ \pi_{A_i, A} + \rho_1 b_i = \rho_1 \dot{x}_i. \] (12)

On allowing \(1 \rightarrow 2\) in (11) we obtain the Cauchy form of the equations of motion regarding 2 as the current configuration. It is worth noting that the Euler relation
\[ \frac{\partial}{\partial X_A} \left( J \frac{\partial X_A}{\partial x_i} \right) = 0 \] (13)

becomes
\[ (J_{12}(1', 2)), 1' = 0. \] (14)

This reduces to
\[ (1', 2), 1' = 0 \] (15)
in the case of an incompressible body.

The second Piola-Kirchhoff stress, denoted here by \(T_{AB}\) or \(T\), is defined by
\[ T = \pi p^{-T} = J p^{-1} \sigma p^{-T}. \] (16)

1) Without ambiguity we may write the transpose of the inverse of \(p\) as \(p^{-T}\).