Second-Order Differential Equation Associated with Contact Geometry: Symmetries, Conservation Laws and Shock Waves

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Abstract. We describe the structure and the correspondence between contact symmetries and conservation laws for second-order differential equations associated with contact geometry. The construction of the Hugoniot-Rankine conditions on discontinuous solutions (shock waves) in natural terms of symbols and generating functions is given. As an example, we demonstrate our techniques for complete descriptions of the infinite-dimensional algebra of contact symmetries and the infinite-dimensional ideal of conservation laws for the von Kármán equation in gas and hydrodynamics, as well as for the construction of 4-parametric discontinuous solutions (shock waves) with a spiral rotation on boundary surfaces (shock wave front sets).


Key words: nonlinear differential equations, contact geometry, sl(2, R)-representation, divergence type, contact symmetry, conservation law, Hugoniot-Rankine conditions.

0. Introduction

Symmetries and conservation laws are two main characteristics of multivalued solutions of differential equations. In this article, we study the Monge–Ampere equations (in Lychagin’s sense). A new approach to these equations, based on a representation technique, is proposed.

We describe the structure and correspondence between the symmetries and conservation laws for these equations. In particular, for the Monge–Ampere equations, an analog to the inverse Noether theorem is obtained: conservation laws generate a subalgebra in a contact symmetry algebra in the common case and an ideal in this algebra under a divergence type and condition of nondegeneracy. The use of conservation laws allows us to describe conditions on discontinuous solutions (shock waves) in natural terms of symbols and generating functions similar to the classical Hugoniot–Rankine conditions.

As an example, we demonstrate our technique for a complete description of the infinite-dimensional algebras of contact symmetries and conservation laws for the von Kármán equation in gas and hydrodynamics. We also construct discontinuous solutions (shock waves) associated with a 4-parametric set of conservation law
pairs. These conservation laws induce a spiral rotation on the boundary surfaces (shock wave front sets).

1. A Set of Lie Structures over 1-Jet Manifold

Let $M$ be a smooth manifold, $\dim M = n$. Let $\mu_x$ be the ideal of the ring $C^\infty(M)$ associated with a point $x \in M$: $\mu_x = \{f \in C^\infty(M) \mid f(x) = 0\}$. A smooth fiber bundle $\pi_k: J^k M \to M$ with the fiber $J^k_x M = C^\infty(M)/\mu_x^{k+1}C^\infty(M)$ over a point $x \in M$ is called a $k$-jet fiber bundle. The image of a function $f \in C^\infty(M)$ in a fiber $J^k_x M$ we denote by $j_k(f)_x = [f]_k^x$. Denote by $J^k(M)$ the module of smooth sections of the fiber bundle $J^k M$. By $S_{j_k(f)} \subset J^k(M)$ we denote the section $S_{j_k(f)}(m) = j_k(f)_m, m \in M$.

Any smooth map $F: M_1 \to M_2$ generates a module homomorphism $\mathcal{J}^k(F): \mathcal{J}^k(M_2) \to \mathcal{J}^k(M_1), [f]_{m_2}^k \mapsto [F^*(f)]_{m_1}^k$, where $f \in C^\infty(M_2), m_1 \in M_1, m_2 \in M_2, m_2 = F(m_1)$.

Recall some basic results on the geometric structure on $J^1 M$ [1].

**Proposition 1.1.** There exists a unique element $\rho_1 \in J^1(J^1 M)$ such that for any $\theta \in J^1(M)$ we have $J^1(\theta)(\rho_1) = \theta$.

The module $J(M)$ is the direct sum of the module of 1-forms $\Lambda^1(M)$ and the ring $C^\infty(M)$: $J^1(M) = \Lambda^1(M) \oplus C^\infty(M)$. Therefore, any element $s \in J^1(M)$ may be understood as a pair $(\omega, f), \omega \in \Lambda^1(M), f \in C^\infty(M)$.

Define an operator

$$D: \mathcal{J}^1(M) \to \Lambda^1(M)$$

$$(\omega, f) \mapsto df - \omega,$$

where $d$ is the de Rham operator.

**Proposition 1.2.** The fiber bundle $J^1 M$ possesses a natural contact structure defined by the universal Cartan 1-form $U_1 = D\rho_1$.

A distribution $K: x_1 \mapsto \text{Ker} \ U_{1,x_1}, x_1 \in J^1 M$, is called the Cartan distribution.

Denote by $C^\infty_x(J^1 M)$ the set of functions $\alpha \in C^\infty(J^1 M), \alpha \neq 0$ at any point $x_1 \in J^1 M$. We also define $U_1^\alpha = (1/\alpha)U_1, \alpha \in C^\infty_x(J^1 M)$.

**Definition 1.1.** A differential ideal $\mathcal{C}$ in the algebra $\Lambda^*(J^1 M)$ generated by the form $U_1$ is called the Cartan ideal.

It is obvious that any form $U_1^\alpha, \alpha \in C^\infty_x(J^1 M)$, is a generator of the Cartan ideal $\mathcal{C}$ as well.