Abstract. In this article, we review the construction of Hamiltonian perturbation theories with emphasis on Hori’s theory and its extension to the case of dynamical systems with several degrees of freedom and one resonant critical angle. The essential modification is the comparison of the series terms according to the degree of homogeneity in both $\sqrt{\varepsilon}$ and a parameter which measures the distance from the exact resonance, instead of just $\sqrt{\varepsilon}$.

1. Introduction. Formal Averaging Theories

The construction of theories aiming at a precise description of the celestial motions is an old problem in astronomy. The complexity of the equations of the N-planets problem ($N \geq 2$) has been overcome by astronomers through very elaborate techniques. Most of these theories were founded on the algebraic skills of some astronomers and classical treatises on perturbations theory assemble a large number of algebraic tricks used to reach the proposed targets. We could quote Sampson’s theory of the four Galilean satellites of Jupiter as a milestone in this direction with its mobile frames and “completed time”. These theories were generally good enough for the construction of ephemerides of a rather good quality for the needs of that time. However, almost all attempts of extending them by increasing the order of approximation were frustrated by insurmountable difficulties, notwithstanding the possibilities of using computer algebra in order to expedite the calculations and to avoid errors in the algebraic developments.

Among the techniques devised in the past, Delaunay’s theory of the motion of the Moon is an exception. Instead of looking for tricks to solve every difficulty, Delaunay proposed a well organized iterative procedure which is a paradigm of what is done nowadays (see Brouwer and Clemence, *Celestial Mechanics and Dynamical Astronomy* 66: 39–50, 1997. ©1997 Kluwer Academic Publishers. Printed in the Netherlands.)
He first noted that the equations of variation of the elements used by Lagrange and Laplace could be written in a very simple form by using a special set of variables, usually called Delaunay's variables:

\[
\begin{align*}
\ell &= \text{mean anomaly} \\
\phi &= \text{argument of the pericenter} \\
h &= \text{longitude of the node}
\end{align*}
\]

\[
L = \sqrt{\mu a} \\
G = L\sqrt{1 - e^2} \\
H = G\cos i
\]

where \(\mu\) is the product of the gravitational constant and the mass of the central body, \(a\) the semi-major axis, \(e\) the orbital eccentricity and \(i\) the inclination of the orbit over the reference plane.

With these variables, the equations of variation of the elements are the Delaunay equations

\[
\begin{align*}
\frac{d\ell}{dt} &= \frac{\partial F}{\partial L} \\
\frac{d\phi}{dt} &= \frac{\partial F}{\partial G} \\
\frac{dh}{dt} &= \frac{\partial F}{\partial H}
\end{align*}
\]

\[
\begin{align*}
\frac{dL}{dt} &= -\frac{\partial F}{\partial \ell} \\
\frac{dG}{dt} &= -\frac{\partial F}{\partial \phi} \\
\frac{dH}{dt} &= -\frac{\partial F}{\partial h}
\end{align*}
\]

\[
F = -\frac{\mu^2}{2L^2} + \varepsilon R(L, G, H, \ell, \phi, h)
\]

\(R\) is the potential of the disturbing forces expressed in terms of the Delaunay variables, written here as a time independent function only for simplicity, and \(\varepsilon\) is a small parameter of the order of the relative value of the disturbing masses. The variational equations are in canonical form. Delaunay then introduced his 'operation' and performed it successively many hundreds of times. Delaunay's operation starts with the choice of one trigonometric term in the Fourier expansion of \(R\), say

\[
W_1 = A_1(L, G, H)\cos(k_1'\ell + k_1''g + k_1'''h)
\]

and the consideration of the dynamical system defined by the abridged Hamiltonian

\[
F^{(1)} = -\frac{\mu^2}{2L^2} + \varepsilon W_1.
\]

This system is integrable and the main step of Delaunay's operation is to obtain one solution of this abridged system and to use it to derive one canonical transformation leading to the elimination of the term \(W_1\) from the given system [in fact, the transformation achieves the substitution of this term by another one with a coefficient of the order of \(O(\varepsilon^2)\)]. After eliminating \(W_1\), one may choose another term, \(W_2\), and repeat the operation. This operation is repeated as many times as possible.