A DYNAMICAL EQUIVALENT TO THE EQUILATERAL LIBRATION POINTS OF THE EARTH-MOON SYSTEM

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Abstract. Consider the Earth-Moon-particle system as a Restricted Three Body Problem. There are two equilateral libration points. In the actual world system, those points are no longer relative equilibrium points mainly due to the effect of the Sun and to the noncircular motion of the Moon around the Earth. In this paper we present the problem as a perturbation of the RTBP and we look for the dynamical equivalent of $L_{4,5}$. It turns out to be a quasiperiodic orbit. It is obtained for a simplified model but the procedure to obtain it is general and can be carried out with an additional computational effort.

Key words: Quasiperiodic perturbations – quasiperiodic solutions – libration points – algebraic manipulators

1. Introduction

Relative equilibrium solutions or libration points are well known in Celestial Mechanics. They or nearby orbits can be useful for space missions. However it turns out that the actual solar system is more complex. We can define geometrical libration points. For instance, for the Earth-Moon system we can define points $L_{4,5}$ which belong to the instantaneous plane of motion of the Moon around the Earth and such that the distances to Earth and to Moon are equal to the actual distance from the Moon to the Earth. The effects of the remaining bodies, specially the Sun, and the noncircular (even non elliptical!) motion of the Moon around the Earth, prevent this point to be a relative equilibrium one. Here we write the full problem as a perturbation of the RTBP. We look for a dynamical equivalent of the libration points, that is, for a solution of the equations of motion which has, as basic frequencies, the ones of the perturbing bodies. This can be done in several ways. The more general one consists in taking the Hamiltonian with quasiperiodic time-dependent coefficients, then performing canonical time-dependent transformations ignoring (to some order) the temporal dependence and looking for the equilibrium point of the transformed (autonomous up to some
order) Hamiltonian system. Another way consists of looking directly for the solutions as a quasiperiodic function of time with basic frequencies the ones of the coefficients of the equations. The first method is more general allowing to compute, not only the equivalent of the libration points but also the equivalent of the periodic orbits and the tori which are found around \( L_{4,5} \) in the spatial RTBP. However we have chosen the second approach which is more direct. In the subsequent sections we present the equations of motion, the simplified model used in this paper, the related expansions, the method used to obtain an approximate quasiperiodic solution and the corresponding manipulator. We end with the results obtained and a comparison against direct numerical integration of the model starting at the same point that the semianalytical solution. The local behaviour around that solution is also discussed showing that it appears to be mildly unstable.

Previous studies concerning this topic can be found in (Tapley, Schultz, 1970) where only numerical simulations are presented. An analytical approach can be found in (Kamel, Breakwell, 1970). Partial results can also be found in (Gómez, Llibre, Martínez, Simó, 1987), work done under ESOC contract 6139/84/D/JS(SC). If we consider only the Elliptic Restricted Three Body Problem for the Earth-Moon system skipping the influence of the Sun there are some results. The earliest ones can be found in (Szebehely, 1967), p. 599.

2. Equations of motion

It is known that, taking a system of reference with the origin at the center of masses of the solar system and axes parallel to the ecliptic ones, the equation of the motion are:

\[
\ddot{\mathbf{R}} = \sum_{A \in \{S, E, M, P_1, \ldots, P_k\}} \frac{G M_A (\mathbf{R}_A - \mathbf{R})}{|\mathbf{R}_A - \mathbf{R}|^3},
\]

where \( G \) is the gravitational constant, \( \mathbf{R} \) the position vector of the particle, \( \mathbf{R}_A \) the position vector of the body of mass \( M_A \), and \( A \) ranges over the Sun, Earth, Moon and the planets. We ignore all the non Newtonian forces and those not coming from the solar system.

However, this system of reference is not the best for this problem. Let us consider instead a system of reference with the origin in one of the instantaneous equilateral points and the axes defined by the unit vectors \( \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 \) given as follows:

\[
\mathbf{e}_1 = \frac{\mathbf{r}_{EM}}{|\mathbf{r}_{EM}|}, \quad \mathbf{e}_3 = \frac{\mathbf{r}_{EM} \wedge \dot{\mathbf{r}}_{EM}}{|\mathbf{r}_{EM} \wedge \dot{\mathbf{r}}_{EM}|}, \quad \mathbf{e}_2 = \mathbf{e}_3 \wedge \mathbf{e}_1,
\]

where \( \mathbf{r}_{EM}(t) \) is the position vector of the Moon with respect to the Earth.