Abstract. The problem of finding a global solution for systems in celestial mechanics was proposed by Weierstrass during the last century. More precisely, the goal is to find a solution of the n-body problem in series expansion which is valid for all time. Sundman solved this problem for the case of $n = 3$ with non-zero angular momentum a long time ago. Unfortunately, it is impossible to directly generalize this beautiful theory to the case of $n > 3$ or to $n = 3$ with zero-angular momentum.

A new ‘blowing up’ transformation, which is a modification of McGehee’s transformation, is introduced in this paper. By means of this transformation, a complete answer is given for the global solution problem in the case of $n > 3$ and $n = 3$ with zero angular momentum.

Key words: N-body problem – blowing up transformation – analytic continuation

1. Introduction

(1) Consider $n$ particles moving in a Euclidean 3-space. Let $m_i$, $q_i$ be the mass and the position of the $i$-th particle, respectively, and let $t$ represent the time. The potential function of this gravitational system is

$$U = \sum_{1 \leq i < j \leq n} \frac{m_im_j}{|q_i - q_j|}$$

and the equations of the motion are

$$q''_i = \sum_{j=1}^{n} \frac{m_im_j(q_i - q_j)}{|q_i - q_j|^3} = \frac{\partial U}{\partial q_i} \quad (i = 1, 2, ..., n).$$

Introduce the $3n \times 3n$ mass matrix $m = \text{diag}(m_1, m_1, m_1; \ldots; m_n, m_n, m_n)$ and take the $3n$-vectors

$$q = (q_1, q_2, \ldots, q_n)^T; \quad p = (p_1, p_2, \ldots, p_n)^T$$

as coordinates of phase space. The equations of motion now are expressed as

$$q' = M^{-1}p; \quad p' = \nabla U(q) \quad (1)$$

* The main result in this paper has appeared in Chinese in Acta Astro. Sinica. 26 (4), 313–322. In this version some mistakes have been rectified and the problems we solved are now expressed in a much clearer fashion.
where $\nabla U(q)$ represents the gradient vector of the potential function $U$ with respect to $q$.

Because the potential function $U = \infty$ when $q_i = q_j \ (i \neq j)$, Equation (1) is an analytic vector field only on

$$\hat{R}^{6n} = \{R^{3n} \setminus \Delta \} \times R^{3n}$$

where $\Delta = \bigcup \Delta_{ij}$

$$\Delta_{ij} = \{(q_1, q_2, ..., q_n)^T ; q_i = q_j \} .$$

**DEFINITION 1.** For any given initial condition $t_0, (q_0, p_0) \in \hat{R}^{6n}$, we call the maximum solution obtained by analytic extension passing through $(q_0, p_0)$ a **global solution**. The corresponding $t$ interval of this solution is called the **time interval of existence** of this solution.

**DEFINITION 2.** A solution is **well-behaved** if $R$ is its interval of existence. Otherwise, we say the solution is **singular**.

Since we are concerned with series expansion, $t$ will be considered as a complex variable and the global solution is the solution extended along the real axis of $t$.

In this paper, we do not consider 'motion after a singular point' or even 'motion after a binary collision'. Although with regularization one can define 'motion after a binary collision', the regularized system can admit other singular solutions for which the concept 'motion after stop time' does not make sense. (See for example, Mather-McGehee's paper [4], where regularized collisions accumulate at a limit point.) Therefore, what is given here is a series solution of Equation (1) which is valid for each global solution.

(2) If the global solution starting from the given initial state is well-behaved, the Cauchy existence theorem affirms that for every real $t$ one can find a complex neighbourhood $W^t$ of $t$ such that the solution is analytic on $W^t$. The union of all these $W^t$ is an open neighbourhood of the real axis where the solution is analytic. Furthermore, if one finds the conformal mapping which maps this domain onto the unit disk (we know such a mapping always exists) and expands $(q, p)$ in the new complex variable, then $(q, p)$ will converge on the unit disk. This is the series expansion we seek.

If the solution is singular, this scheme also makes sense. There exists an analytic domain containing the maximal time interval of existence. Then, if one finds the conformal mapping which maps this domain to the unit disk, one obtains a series expansion which is valid for all the real values of the solution. To follow this scheme, one should answer the following questions: