Determinants of Cauchy–Riemann Operators as $\tau$-Functions

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Abstract. The $\tau$-functions introduced by Sato, Miwa, and Jimbo for the deformation theory associated with the Riemann–Hilbert problem on $\mathbb{P}^1$ is shown to be a determinant for a singular Cauchy–Riemann operator whose domain incorporates functions with prescribed branching behavior. The analysis relies heavily on previous work of Malgrange on monodromy preserving deformation theory.


Keywords. Cauchy–Riemann operators, $\tau$-functions.

0. Introduction

The object of this paper is to give an interpretation of the $\tau$-functions introduced by Sato, Miwa, and Jimbo for monodromy preserving deformations of the Cauchy–Riemann equations on $\mathbb{P}^1$ [10, II] as the determinant of an associated Cauchy–Riemann operator on the spin bundle over $\mathbb{P}^1$. The monodromy in the problem is now reflected by prescribed branch discontinuities in the domain of the associated 'Cauchy–Riemann' operator. This point of view is at least implicit in the fifth paper [10, V] on Holonomic quantum fields and it is also implicit in the notion of vertex insertions in conformal field theory [2].

One motivation for our desire to reinterpret the $\tau$-function as the determinant of a Cauchy–Riemann operator is that there are natural situations in which one has a '$\tau$-function' but there is no clear analogue of the linear differential equations that are the arena for deformation theory. An example is provided by the correlations of the two-dimensional Ising model on a lattice. These correlation functions are naturally determinants (more precisely Pfaffians) of certain inhomogeneous difference operators on $l^2(\mathbb{Z}^2)$ [7] but it is not terribly natural to formulate monodromy preserving deformation theory on the lattice. On the other hand, it was the discovery that a certain continuum scaling limit of the two point Ising correlation could be expressed in terms of Painlevé transcendent by Wu, McCoy, Trace, and Barouch in [14] that blossomed into a general theory of monodromy preserving deformations and associated $\tau$-functions in the series of papers [10, I–V]. In this theory, the scaled correlations of the Ising model are the prototypical example of a $\tau$-function. In
studying the scaling limit of the two-dimensional Ising model it is useful to have a picture in which the nature of the \( \tau \)-function does not change as one passes from the lattice to the continuum.

In [4], Malgrange gives a very elegant geometric interpretation for the \( \tau \)-function that arises for monodromy preserving deformations of the Cauchy–Riemann equations as part of his analysis of the deformation equations themselves. Another reason for our interest in reinterpretting the \( \tau \)-function comes from the desire to generalize this work of Malgrange to other settings. It was not immediately clear to us from the work of Sato, Miwa, and Jimbo or Malgrange what the appropriate definition of a \( \tau \)-function should be for a Riemann surface different from \( \mathbb{P}^1 \). This paper does not answer this question but I believe it does at least give a clear formulation of the problem to be solved.

There is a also a difficulty in generalizing the Malgrange analysis to the Euclidean Dirac equation. In Malgrange the \( \tau \)-function is understood to define a divisor (in the ‘space of branch points’) where an auxiliary family of holomorphic line bundles on \( \mathbb{P}^1 \) is holomorphically nontrivial. There are no holomorphic bundles evident in the Euclidean Dirac problem but the notion that the \( \tau \)-function is the determinant of the Euclidean Dirac operator with a domain incorporating specified branching does carry over. We hope to return to the problem of generalizing the Malgrange analysis to the Euclidean Dirac equation in another place.

This paper is organized in four sections. The first section deals with determinant bundles for Cauchy–Riemann operators on the spin bundle on \( \mathbb{P}^1 \) following the ideas of Quillen [9]. A localization of the determinant bundles is introduced following some ideas in Witten [13] and this is shown to lead to a mapping into the \( \text{det}^* \) bundle on the boundary of the localization as defined by Segal and Wilson [11]. Although our main interest in this paper is in singular Cauchy–Riemann operators, for simplicity we treat only smooth operators in this first section. The reader should be aware that in referring to singular Cauchy–Riemann operators we step outside the bounds of convention. In as much as Cauchy–Riemann operators define complex structures on \( C^\infty \) vector bundles they are always smooth operators. Still, we believe that the reason for referring to the operators that we introduce as singular Cauchy–Riemann operators will be apparent to the reader as soon as they are introduced. Our strategy in dealing with these singular Cauchy–Riemann operators will be to localize them away from their singularities.

The second section presents an account of a connection in a group of bundle automorphism of the \( \text{det}^* \) bundle that will be used in trivializing the determinant bundle we are interested in. The connection one form of this connection relative to the canonical section is the regularization of the logarithmic derivative of the determinant of a Toeplitz operator introduced by Malgrange [4].

The third section introduces the singular Cauchy–Riemann operators of interest to us and reviews their relation to the Riemann–Hilbert problem. We also begin the translation of the Malgrange analysis into this setting.

The fourth section is largely devoted to connecting the analysis of the appropriate