Nonlinear boundary value problems for the annular membrane:
A note on uniqueness of positive solutions

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Abstract. The study of axisymmetric deformations of annular membranes under normal surface loads within the framework of the Föppl-Hencky small finite-deflection theory is continued after progress in this field has been made in the recent work of Grabmüller and Weinitschke [5]. When a radial displacement is applied at the inner edge and a radial tension or a displacement at the outer edge, the mathematical question of uniqueness of tensile solutions of the resulting nonlinear boundary value problems has not been settled yet. In this paper, uniqueness is proved in a parameter range of the boundary conditions, where previous methods have failed.

1. Introduction

We consider annular elastic membranes under the action of axisymmetric surface loads \( p(r) \), within the small finite deflection theory of Föppl-Hencky [1,2], first investigated by Schwerin [3]. The axisymmetric stresses and displacements are determined by the solution of the single differential equation

\[
y'' + \frac{3}{x}y' + \frac{2}{y^2}R^2(x, \epsilon) = 0, \quad 0 < \epsilon < x < 1, \quad x = \frac{r}{a}
\]

\[
R(x, \epsilon) := \frac{2}{x^2} \int_{\epsilon}^{x} t\bar{p}(t) \, dt, \quad \bar{p} := \frac{p(t)}{p_0}, \quad p_0 := \max |p(t)|
\]

satisfying appropriate boundary conditions. The notation here is the same as in an earlier paper on the subject [5]. In particular, the radial stress \( \sigma_r \) and the radial displacement \( u \) are given, in terms of \( y(x) \), by

\[
\sigma_r = c_1 y(x), \quad u = c_2 x (xy'(x) + (1 - \nu)y(x)),
\]

with certain constants \( c_i > 0 \). We refer the reader to [4] and [5] for more details and for an account of earlier work on this problem.

Physical reasoning suggests to prescribe \( \sigma_r \) or \( u \) at the inner and outer edges. Accordingly, we are concerned with the following four boundary conditions

\[
y(\epsilon) = s \quad \text{or} \quad \epsilon y'(\epsilon) + (1 - \nu)y(\epsilon) = h
\]
at the inner edge $x = \epsilon \ (r = r_i)$ and

$$y(1) = S \quad \text{or} \quad y'(1) + (1 - \nu)y(1) = H \quad (1.3)$$

at the outer edge $x = 1 \ (r = a)$. Here $\nu$ denotes Poisson’s ratio, $0 \leq \nu \leq 0.5$, $\epsilon > 0$ is the scaled inner radius of the annulus. Furthermore, we assume $S > 0$ and $s \geq 0$, as we shall restrict attention to tensile solutions $y(x) \geq 0$. Combining, respectively, one condition from (1.2) and one condition from (1.3) with the differential equation (1.1), we are led to consider four different boundary value problems (BVP’s). In the sequel, we shall refer to these BVP’s as problems $(s, S)$, $(s, H)$, $(h, S)$, or $(h, H)$, with obvious notation.

Attempts have been made in the past to solve the mathematical problems of existence and uniqueness of solutions of the four BVP’s (1.1)–(1.3). In [4], Weinitschke proved existence and uniqueness of tensile solutions for a restricted range of the parameters $s$, $h$, $S$ and $H$. More recently, the problems $(s, S)$ and $(s, H)$ were completely solved by Grabmüller and Weinitschke [5] for $S > 0$, $H \in \mathbb{R}$ and uniform load, provided $s = 0$. For $s > 0$, problems $(s, S)$ and $(s, H)$ were solved completely for arbitrary surface load except that in Problem $(s, H)$ the values of $s$ and $H$ had to satisfy the inequality $2s + H(1 - c^2)/c^2 > 0$ in the existence part of the problem [5]. In that paper it was also observed that uniqueness of solutions for problems $(h, S)$ and $(h, H)$ cannot be proved by the standard arguments used in problems $(s, S)$ and $(s, H)$. On the other hand, uniqueness can be expected on physical grounds, so long as the boundary parameters are in a range where only tensile solutions exist, for example if $h \leq 0$ and $S > 0$ in Problem $(h, S)$.

It is the purpose of this paper to supply the outstanding uniqueness proofs for both problems $(h, S)$ and $(h, H)$. Our method consists in transforming each BVP into a more tractable form such that the well-known maximum principle of Hopf can be used conveniently. Throughout the paper we admit arbitrary axisymmetric surface loads, which should not vanish identically.

The question of existence of tensile solutions of the problems $(h, S)$ and $(h, H)$, which was not considered in [5], will be treated in a forthcoming paper [7], where a considerable extension of earlier results will be achieved.

2. The uniqueness of positive regular solutions

We shall be concerned with strictly tensile regular solutions of the BVP’s $(h, S)$ and $(h, H)$, that is, with positive and regular solutions $y(x)$ of Eq. (1.1) satisfying

(i) $y(x) > 0$ for all $x \in [\epsilon, 1]$, (ii) $y \in C^2(\epsilon, 1) \cap C^1[\epsilon, 1]$.

Recall that Problem $(h, S)$ means that a (dimensionless) radial displacement $h \in \mathbb{R}$ is prescribed at the inner, and a radial stress $S > 0$ at the outer edge.

1 This restriction has been removed in [7].