Abstract. We investigate the escape regions of a quartic potential and the main types of irregular periodic orbits. Because of the symmetry of the model the zero velocity curve consists of four symmetric arcs forming four open channels around the lines $y = \pm x$ through which an orbit can escape. Four unstable Lyapunov periodic orbits bridge these openings.

We have found an infinite sequence of families of periodic orbits which is the outer boundary of one of the escape regions and several infinite sequences of periodic orbits inside this region that tend to homoclinic and heteroclinic orbits. Some of these sequences of periodic orbits tend to homoclinic orbits starting perpendicularly and ending asymptotically at the $x$-axis. The other sequences tend to heteroclinic orbits which intersect the $x$-axis perpendicularly for $x > 0$ and make infinite oscillations almost parallel to each of the two Lyapunov orbits which correspond to $x > 0$ or $x < 0$.

1. Introduction

We consider the dynamical system of two degrees of freedom

$$V = \frac{1}{2}(x^2 + y^2) - \varepsilon x^2 y^2$$

This model has closed zero velocity curves (ZVC) for $\varepsilon < \varepsilon_{\text{esc}} = (4h)^{-1}$ where $h$ is the total energy. For $\varepsilon > \varepsilon_{\text{esc}}$ the ZVC consists of four branches forming four channels (or legs) through which an orbit can escape to infinity. Across these channels there exist four unstable Lyapunov periodic orbits (see Moser, 1958) each of them crossing perpendicularly one of the lines $y = \pm x$ and connecting two opposite branches of the ZVC.

The potential (1) has been mentioned by Hénon and Heiles (1964), by Andrle (1979) and it is a special case of the Verhulst potential (1979). Churchill, Peceli and Rod (1975, 1979) gave some geometric conditions under which each Lyapunov periodic orbit of the above system is unstable and isolated in a very large region in the sense that it is the only bounded orbit of a given energy that remains in this region for all time. Also Churchill and Rod (1980) prove the existence of nondegenerate homoclinic and heteroclinic orbits to these Lyapunov orbits under some general conditions. Robe (1986, 1987) studied the periodic orbits and their stability using a three dimensional model that represents the inner part of a triaxial galaxy.

Here we study numerically this system in order to find the structure of its escape regions. We take the total energy constant and equal to $h = 0.120$ and we consider
coupling $\varepsilon$ as a parameter. The only simple periodic orbits for $\varepsilon < \varepsilon_{\text{esc}} = 2.0833$ are:

(i) the two axes of symmetry which are stable for $\varepsilon < 4.8473$

(ii) the straight lines $y = \pm x$ which exist only for $\varepsilon < \varepsilon_{\text{esc}}$. The stability and the bifurcations of these orbits are discussed in the Appendix.

(iii) a closed curve around the origin crossing perpendicularly the axes of symmetry and the lines $y = \pm x$; this periodic orbit is unstable for every value of $\varepsilon$.

We consider only orbits starting perpendicularly to the $x$-axis with $y > 0$. Therefore for any given $\varepsilon$ the initial conditions are given by $x$. The initial values for $x$ are in the interval $-2h \leq x \leq 2h$. We call the escape-$n$ ($\text{Esc-}n$) region any connected region of the $(\varepsilon, x)$ plane each point of which represents an orbit escaping to infinity after $n$ crossings of the $x$-axis where the counting of crossings includes the initial point on the $x$-axis. We confine our study to the Esc-1 regions, namely to orbits which escape to infinity without crossing again the $x$-axis, besides the initial starting point.

In the case of the model system

$$V_1 = \frac{1}{2}(x^2 + y^2) - \varepsilon xy^2 + ae^2 y^4$$

(2)

we have found (Barbanis, 1986) that the Esc-1 region is surrounded by an infinite sequence of triplets. Each triplet consists of three families of simple periodic orbits defined by three characteristic curves $x = x(\varepsilon)$. These orbits cross the $x$-axis perpendicularly; the orbits of the two families at one point and the orbits of the third family at two points. All these orbits are irregular periodic orbits (IPOs). By IPO we mean an orbit which does not belong to a family of the unperturbed system ($\varepsilon = 0$) or to any of its bifurcations. The characteristic curves $x = x(\varepsilon)$ exist all the way from $\varepsilon = \varepsilon_{\text{min}}$ to $\varepsilon = \infty$ if the Esc-1 region extends to infinity, as it happens in the case $a = 0$ (Contopoulos, 1981) or to $\varepsilon = \varepsilon_{\text{max}}$ if the Esc-1 region is finite as it happens when $a \neq 0$ (Barbanis, 1986).

On the other hand we have found a finite number of closed triplets which do not surround a Esc-1 region (Barbanis, 1986). These triplets exist in a narrow interval of values of $a$. As $a$ tends to the lower limit of this interval the number of triplets tends to infinity, while as $a$ tends to the upper limit the number of triplets tends to zero.

Font and Grau (1988) have considered the model (2) when $\varepsilon = 1$ and they have studied, for some critical values of $a$, the characteristic curves $x = x(h)$ of triple periodic orbits which intersect the axis $y = 0$ at a point $x$ with $\dot{x} = 0$. Their study allowed them to explain in a geometric way the results found by Barbanis (1985) when $a = 0.5$.

Detailed scanning in $x$ of the Esc-1 region of the model system (2) for several values of $\varepsilon$ showed that all orbits are escape-1 orbits. Therefore it was a surprise to find that in the case of the system (1) the region of the $(\varepsilon, x)$ plane, which is surrounded by an infinite sequence of families of IPOs is not an Esc-1 region for every possible value of $x$ for a given $\varepsilon$. On the contrary, this region is traversed by many sequences of families of IPOs. These sequences relate to homocline and heteroclinic points of the system (1). The study of the Esc-1 region of this system is the purpose of this paper.