RELATIVISTIC EFFECTS IN THE
MOTION OF ARTIFICIAL SATELLITES:
THE OBLATENESS OF THE CENTRAL BODY II

JOACHIM HEIMBERGER, MICHAEL SOFFEL AND HANNS RUDER

Lehrstuhl für Theor. Astrophysik der Universität Tübingen
Auf der Morgenstelle 12, D-7400 Tübingen, FRG
E-mail PEH1001@DTUPEVSA.

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Abstract. The motion of artificial satellites in the gravitational field of an oblate body is discussed in the post-Newtonian framework using the technique of canonical Lie transformations. Two Lie transformations are used to derive explicit results for the long-periodic and secular perturbations for satellite orbits in the Einstein case.

Keywords: PN-Theory, oblate body, Lie transformations, longperiodic perturbations

1. Introduction

The post-Newtonian (PN) acceleration of a satellite in standard PN-coordinates in the gravitational field of a non-rotating central body is given by (e.g. Will 1981)

\[ \frac{d\mathbf{v}}{dt} = \nabla U + \frac{1}{c^2} \left[ -2\nabla U^2 - 4\mathbf{v} \cdot (\nabla \mathbf{v}) U + \mathbf{v}^2 \nabla U \right] \]  

Mainly because of the complexity of expressions encountered in the following derivations, here we will consider the Einstein PN case only (PPN parameters $\beta = \gamma = 1$) in contrast to the first part of this article (Soffel et al. 1988, called (I) in the following), where $\beta$ and $\gamma$ were explicitly taken into account.

In (I) we have analyzed those PN-perturbations of satellite orbits that are induced by the $J_2c^{-2}$ terms on the r.h.s. of (1), where $J_2$ is the quadrupole moment of the central body, by means of the $S, T, W$- version of the perturbation equations for Keplerian (osculating) elements to first order. As we have already remarked in (I) there are additional second order mixed perturbations due to the Newtonian quadrupole field and the Schwarzschild acceleration that are of the same order of magnitude that have not been treated in (I), but will be discussed below. To this end we employ the technique of canonical Lie transformations as introduced by Hori (Hori 1966) and Deprit (Deprit 1969; see also Mersman 1970). This technique starts from the (specific) Lagrangian $\mathcal{L}$ corresponding to (1) (e.g. Will 1981)

\[ \mathcal{L} = \frac{1}{2} \mathbf{v}^2 + U + \frac{1}{c^2} \left[ \frac{1}{8} \mathbf{v}^4 - \frac{1}{2} U^2 + \frac{3}{2} U \mathbf{v}^2 \right] \]  

giving the canonical momentum $p$ and Hamiltonian $\mathcal{H}$ as

$$p = \frac{\partial L}{\partial v} = \left[ 1 + \frac{1}{2} \frac{v^2}{c^2} + \frac{3}{c^2} \right] v$$

(3)

and

$$\mathcal{H} = vp - L = \frac{1}{2} p^2 - U + \frac{1}{c^2} \left[ -\frac{1}{8} p^4 + \frac{1}{2} U^2 - \frac{3}{2} U p^2 \right].$$

(4)

Neglecting all deviations from axial symmetry, higher multipole moments of the field generating source and $J_2$-terms (whose consequences for satellite orbits in the Newtonian framework are e.g. discussed in Deprit and Rom 1970) the Hamiltonian takes the form ($\mu = GM$)

$$\mathcal{H} = \frac{1}{2} p^2 - \frac{\mu}{r} + J_2 \frac{\mu R^2}{r^3} \left( \frac{3}{2} \sin^2 \delta - \frac{1}{2} \right)$$

$$+ \frac{1}{c^2} \left[ -\frac{1}{8} p^4 + \frac{1}{2} \frac{\mu^2}{r^2} - \frac{3}{2} \frac{\mu}{r} p^2 \right]$$

$$+ \frac{J_2 R^2}{c^2} \left( \frac{3}{2} \sin^2 \delta - \frac{1}{2} \right) \left( -\frac{\mu^2}{r^4} + \frac{3}{2} \frac{\mu}{r} p^2 \right)$$

or expressed in terms of canonical Delaunay-variables $l, g, h, L, G, H$, defined in the usual way by $l = M, L = n a^2 = \sqrt{\mu a}, g = \omega, G = n a^2 \sqrt{1 - e^2}, h = \Omega, H = n a^2 \sqrt{1 - e^2} \cos i$:

$$\mathcal{H} = -\frac{\mu^2}{2L^3} + J_2 \frac{\mu^2 R^2}{L^6} \left( \frac{a}{r} \right)^3 \left[ \frac{1}{4} - \frac{3}{4} \frac{H^2}{G^2} - \frac{3}{4} \left( 1 - \frac{H^2}{G^2} \right) \cos 2(f + g) \right]$$

$$- \frac{\mu^4}{c^2 L^4} \left[ \frac{3}{4} (\frac{a}{r})^2 - \frac{2}{r} \frac{a}{r} + \frac{1}{8} \right]$$

$$+ \frac{J_2 \mu^2 R^2}{c^2 L^6} \left[ \frac{1}{4} - \frac{3}{4} \frac{H^2}{G^2} - \frac{3}{4} \left( 1 - \frac{H^2}{G^2} \right) \cos 2(f + g) \right] \left[ 2 \left( \frac{a}{r} \right)^4 - \frac{3}{2} \left( \frac{a}{r} \right)^3 \right].$$

(5)

Here, $f$ and $a/r$ have to be considered as functions of $l, L$ and $G$.

Since we are not interested in the terms of higher order in $J_2$, we write

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_1 + \frac{1}{c^2} \mathcal{H}_2$$

(6)

with $\mathcal{H}_0 = -\mu^2/2L^2$ and $\mathcal{H}_2/2$ is given by the $J_2 c^{-2}$ term in (5). We now want to solve the canonical equations of motion for the Delaunay variables by means of canonical Lie transformations of the form

$$(x, X) = E_w(y, Y),$$

(7)

† Since $p = v + O(c^{-2})$ the corresponding osculating elements differ from Keplerian osculating elements by terms of order $c^{-2}$. E.g. $p^2/2 - \mu/r = -\mu/2a = -\mu^2/2L^2$ etc.

‡ For the Earth $J_2 \sim 10^{-3}, Uc^{-2} \sim 10^{-9}$; hence, the $J_2 U/c^2$ terms are of the same order as the Newtonian $J_2$-terms to fourth order.