An integral equation formulation of three dimensional anisotropic elastostatic boundary value problems

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ABSTRACT

An approximate solution capability is developed to handle three dimensional anisotropic elastostatic boundary value problems. The method depends crucially on the existence and explicit definition of a fundamental solution to the governing partial differential equations. The construction of this solution for the anisotropic elastostatic problem is presented as is the derivation of the expression for the surface tractions necessary to maintain the fundamental solution in a bounded region. After the fundamental solution and its associated surface tractions are determined, a real variable boundary integral formula is generated which can be solved numerically for the unknown surface tractions and displacements in a well-posed boundary value problem. Once all boundary quantities are known, the field solution is given by a Somigliana type integral formula. Techniques for numerically solving the integral equations are discussed.

ZUSAMMENFASSUNG


Introduction

In recent years, accurate approximate solutions have been obtained to a variety of elasticity problems [1, 2, 3, 4, 5, 6] through the use of Somigliana type integral formulas. This approach to the solution of boundary value problems can be outlined briefly as follows: A Somigliana identity provides a formal representation for the displacement of an interior point in the body in terms of integrals involving boundary data. If, in these integrals, the interior point is allowed to approach a boundary point, a limiting form of the Somigliana formula is obtained which provides integral relations between boundary tractions and displacements. In a well-posed boundary value problem the tractions and displacements are not concurrently assigned over the entire surface. The boundary integral equations provide then the necessary relations for determining (approximately) the boundary information which was not prescribed. Once these integral equations have been solved for the unknown data, the Somigliana formula can be employed to obtain the displacement field.

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As is clear from the papers cited above, the method has reached a considerable stage of development and is providing the analyst with a significant alternative to the finite element and finite difference techniques for certain problems. Three features of the method are probably most responsible for its emergence as an important tool in the numerical solution of boundary value problems. First, the integral equations are not expressed in terms of non-physical accessory functions but are written directly in terms of the boundary tractions and displacements. Secondly, the method is not restricted to particularly simple or special boundary geometries; i.e., it is independent of shape. The final feature of note is that numerical approximations are not made over the entire region occupied by the body but are confined to the surface tractions and surface displacements. The attractiveness of these features coupled with the considerable success already achieved with the method motivated an attempt to open up the field of linear three-dimensional anisotropic elastostatics to attack via this procedure. The present analysis is concerned with such an extension of the technique.

In the next section we introduce our notation and record the basic equations pertinent to the integral equation method. A glance at the earlier mentioned papers reveals that the method depends crucially on the existence and explicit definition of a fundamental solution to the appropriate governing partial differential equations. Physically this solution is the displacement field in a body of infinite extent subjected to a concentrated unit body force. The third section of this paper is devoted to establishing this solution for a homogeneous anisotropic elastic medium. The road taken for its determination is the method of decomposition into plane waves used by John [7]. Basically this method of constructing the fundamental solution amounts to reducing the given problem to an intermediate one which, due to the Cauchy-Kowalewski theorem, is known to have a unique solution. Since the equations for homogeneous anisotropic elasticity have constant coefficients, this Cauchy problem can be solved explicitly and the fundamental solution is then obtained by quadrature. This fundamental solution has also been obtained by Synge [8] who used Fourier transforms in its derivation.

As utilized in Somigliana's identity, the fundamental solution represents the displacement field in a body occupying a finite region of space. This solution can be maintained in such a region if, in addition to the concentrated unit body force, appropriate surface tractions are applied at the boundary. These tractions are also a key ingredient in the identity and their derivation is the subject of the fourth section. In the paper's concluding section, the limiting form of the Somigliana identity as an interior point approaches the boundary is established and its numerical solution is discussed.

**Preliminaries. Notation and basic equations**

Consider a body which occupies an open region \( R \) in Euclidean 3-space \( E \) and has a bounding surface \( \partial R \) which is piecewise smooth. We seek the displacement field \( u_i \) which satisfies the system of equations

\[
\epsilon_{ijk} u_k, \mu = 0
\]

(1)

within \( R \) and certain appropriate boundary conditions on \( \partial R \). Equations (1) are the equations of equilibrium in the absence of body forces for a linear elastic solid obeying the constitutive relation

\(^1\) Appendix miniscule Latin subindices have the range \((1, 2, 3)\) and denote the Cartesian components of some entity with respect to a fixed frame of reference. Repeated suffixes are to be summed and partial differentiation is indicated by a comma between suffixes. The range of Greek subscripts is separately indicated.