INTRINSIC ORBITAL COORDINATES
ON A HAMILTONIAN SURFACE

MICHAEL E. HOUGH

Textron Defense Systems, Wilmington, MA 01887, U.S.A.

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Abstract. A reversible dynamical system with two degrees-of-freedom is reduced to a second-order, Hamiltonian system under a change of independent variable. In certain circumstances, the reduced order system may be integrated following an orthogonal curvilinear transformation from Cartesian \(x, y\) to intrinsic orbital coordinates \(\zeta, \eta\). Solutions for the orbit position and true time variables are expressed by:

\[
x = f(\zeta, \eta), \quad y = g(\zeta, \eta), \quad \frac{dt}{d\tau} = \pm \left[ \frac{\partial f}{\partial \zeta} + \frac{\partial f}{\partial \eta} \right]^{1/2} \frac{d\tau}{dt},
\]

where \(U\) is the potential function, and \(\tau\) is the new independent variable. The functions \(f, g\) may be expressed by quadratures when the metric coefficients \(g, f\) are specified. Two second-order, partial differential equations specify \(f, g\) and Hamiltonian \(H\). Auxiliary conditions are needed because the solutions are underdetermined. For example, both sets of curvilinear coordinate lines are orbits when certain dynamical compatibility conditions between \(U\) and \(f\) (or \(g\)) are satisfied. Alternatively, when orbits cross the parametric curves, the auxiliary condition \(g = f\) specifies a conformal transformation, and the partial differential equation for \(\dot{\zeta}\) may be reduced to an ordinary differential equation for the orbit curve. In either case, integrability is guaranteed for Liouville dynamical systems. Specific applications are presented to illustrate direct solution for the orbit (e.g., two fixed centers) and inverse solution for the potential.

A. Introduction

This article is concerned with the integrability of a two degree-of-freedom, reversible dynamical system with equations of motion:

\[
\begin{align*}
\dot{u} &= U_x, \quad \dot{v} = U_y, \\
\dot{x} &= u, \quad \dot{y} = v
\end{align*}
\]

where algebraic subscripts denote partial differentiation. Position components \((x, y)\) and inertial velocity components \((u, v)\) are resolved along inertial Cartesian axes. Gravitational forces are expressed by the gradient of an autonomous potential function \(U(x, y)\). The fourth-order system (1) can be reduced to third order because the energy integral specifies velocity magnitude \(P\) as a function of position along the orbit and a constant parameter \(E\) (the energy):

\[
P^2(x, y; E) = 2[U(x, y) + E]
\]

The reduced order system consists of the kinematic Equations (1.2) and a differential equation for the flight path angle \(\gamma (= \tan^{-1}(v/u)):\)

\[
\begin{align*}
\dot{x} &= P(x, y; E) \cos \gamma, \quad \dot{y} = P(x, y; E) \sin \gamma, \quad \dot{\gamma} = \frac{U_y \cos \gamma - U_x \sin \gamma}{P(x, y; E)}
\end{align*}
\]

By eliminating the time variable, these autonomous equations may be reduced to a second-order equation specifying the orbit function $y = F(x; E)$. The origin of time $t_0$ is an integration constant.

In an earlier publication (Hough, 1988), it was shown that the fourth-order system (1) may be reduced to an autonomous, second-order, Hamiltonian system by changing the independent variable:

$$
\frac{dt}{\zeta} = \frac{d\mathcal{H}}{\zeta}, \quad \zeta = v_x - u_y
$$

$$
\bar{u} = \zeta u = -\mathcal{H}_y, \quad \bar{v} = \zeta v = \mathcal{H}_x, \quad \mathcal{H} \equiv \frac{1}{2}(u^2 + v^2) - U \tag{3}
$$

The normalized velocity field is divergence-free because $\bar{u}_x + \bar{v}_y = 0$. Depending on direction of circulation, vorticity $\zeta$ can be positive or negative, corresponding to the two roots of:

$$
\zeta^2 = \frac{\bar{u}^2 + \bar{v}^2}{u^2 + v^2} = \frac{\mathcal{H}_x^2 + \mathcal{H}_y^2}{2(\mathcal{H} + U)}
$$

since $|\nabla \mathcal{H}| \neq 0$. This formulation is fundamentally different from the classical approach of representing orbital motion on an isoenergetic surface ($\mathcal{H}_x = \mathcal{H}_y = 0$) with irrotational velocity field ($\zeta = 0$).

In the three dimensional space with rectangular coordinates ($x, y, \mathcal{H}$), “vertical” motion (along the $\mathcal{H}$ coordinate) cannot occur because of (2). Individual orbits are level curves $\mathcal{H} = E$ stratified on parallel horizontal planes. $\mathcal{H}$ satisfies a second-order, quasi-linear, partial differential equation:

$$
2(\mathcal{H} + U)(\mathcal{H}_{xx}\mathcal{H}_y^2 - 2\mathcal{H}_{xy}\mathcal{H}_x\mathcal{H}_y + \mathcal{H}_{yy}\mathcal{H}_x^2) + 
(\mathcal{H}_x U_x + \mathcal{H}_y U_y)(\mathcal{H}_x^2 + \mathcal{H}_x^2) = 0 \tag{4}
$$

The general solution $\mathcal{H}(x, y; v, \omega)$ depends on two constants $v$ and the orbital phase angle $\omega$. When (4) has a single-valued, analytic solution, a single parameter $E$ defines a family of orbits. The other constants of integration are adjusted to prescribe an orderly pattern of level curves in configuration space. (i.e., no intersections). For a reversible system, prograde or retrograde orbits can be represented as level curves of the same Hamiltonian function.

The true time variable $t$ is expressed by a quadrature evaluated along the orbit implicitly defined by $\mathcal{H} = E$:

$$
t = t_0 \pm \int_{x_0}^{x} \left[\frac{\mathcal{H}_x^2 + \mathcal{H}_y^2}{2(\mathcal{H} + U)}\right]^{1/2} \frac{dx}{\mathcal{H}_y} \quad (\mathcal{H} = E) \tag{5}
$$

where $t_0$ and $x_0$ are initial conditions. Singularities occur at turning points, where $\mathcal{H}_y = 0$. The algebraic sign is determined by direction of circulation.

An inversion of dependent and independent variables $y = F(x; \mathcal{H})$ reduces (4) to